PROCEEDINGS
OF THE
NATIONAL ACADEMY OF SCIENCES
Volume 28 December 15, 1942 Number 12
Copyright 1943 by the National Academy of Sciences

SEXUAL HORMONES IN ACHLYA. V. HORMONE A', A MALE-SECRETED AUGMENTER OR ACTIVATOR OF HORMONE A

By John R. Raper*

Department of Botany and Bacteriology, Indiana University

Communicated November 2, 1942

The rôle of specific sexual hormones in the sexual reaction of two heterothallic species of *Achlya* has been described in earlier papers of this series.[1] Four specific substances were postulated, two secreted by the ♀, bringing about specific reactions in the ♂, and two secreted by the ♂, inducing specific responses in the ♀. On the basis of evidence available at that time it seemed that the entire sexual reaction was initiated wth the secretion of hormone *A* by the vegetative mycelium of the ♀, which induced the formation of sexual organ initials, antheridial branches, on the ♂. In the course of a more detailed analysis of the response of the ♂ plant to hormone *A* from the ♀,[2] it was found that the vegetative mycelium of the ♂ secreted some factor which profoundly influenced its response to hormone *A* from the ♀. This factor, produced by the vegetative ♂ plant independently of any response to the ♀, will hereafter be referred to as hormone *A'* (*A* prime).

The species and sexual strains used in this work are those previously employed in the work on sexual hormones in *Achlya: Achlya bisexualis* ♀ for the production of hormone *A*,[3] and *Achlya ambisexualis* ♂ as test plants for the assay of hormone *A* and for the detection of hormone *A'*. The method of testing is that previously described[2] and the standard hormone *A* solution is the same as that used throughout the work on the characterization of this hormone.[4] Any modifications of previously described methods or techniques will be described in the body of the paper.

The discovery of hormone *A'* came about quite by accident. In the analysis of the response of the ♂ plant to hormone *A*, a series of experiments was performed in an attempt to determine whether the hormone was used up, i.e., removed from the test solution during the course of the reaction, or whether it produced its effect through the catalytic activation of some existing physiological system, the hormone thus remaining un-

changed and unconsumed in the test solution. The first of these experiments was the repeated quantitative determinations of the hormone *A* content of a 10-ml. sample of standard (6 units/ml.) test solution at successive times with new ♂ plants. The results were unexpected, for instead of the response remaining constant or decreasing there was an *apparent* increase in the amount of hormone *A* during the first day of the experiment. This increased activity reached approximately 300% of that of the sample at the beginning of the experiment and of that of a control series of new samples of hormone *A* solution tested at the same intervals. Toward the end of the second day the reaction decreased somewhat, probably because of the dilution of the hormone *A* solution by water unavoidably carried in with each new pair of ♂ plants. The results of these tests could not be explained at the time.

A second series of experiments to determine the ultimate fate of hormone *A* was then performed. Into 10-ml. samples of standard hormone *A* solution in Petri-plates were placed ♂ test plants with two plants in the first plate, four in the second and so on up to twenty plants in the tenth plate. At the end of two hours the average number of antheridial branches per vegetative hypha was determined for the plants in each sample. On three successive days, October 7, 8 and 9, 1940, the results were the same. In each series the intensity of the response varied directly with the number of test plants in the 10-ml. sample of standard hormone *A* solution. The average responses for the three series are given in table 1.

TABLE 1

MASS EFFECT OF ♂ PLANTS ON THEIR RESPONSE TO HORMONE *A*

Number of plants in tests	2	4	6	8	10	12	14	16	18	20
Average number of antheridial branches	5.2	5.8	9.7	9.8	10.6	13.1	13.7	12.5	13.9	14.1

The results from these tests agree strikingly with those of the previous experiment when, in effect, additional plants were added to the sample.

From these experiments it was obvious that the ♂ plants themselves exerted some mass effect on their response to hormone *A*, and that the intensity of their response depended on some factor in addition to that of hormone *A* concentration. The most probable explanation of this increase seemed to be the secretion by the ♂ plant of a factor which augmented the effect of hormone *A*, or which behaved as an activator for hormone *A*. The secretion of such an active substance, hormone *A'*, has been repeatedly confirmed by experiments extending over a period of two years.

The first tests designed to demonstrate the secretion and activity of an active substance from the ♂ were made as follows. The liquid from a number of 48-hour cultures, each containing ten ♂ plants in 10 ml. of water, was filtered, and this filtrate was used in making up a series of test

solutions containing undiluted ♂ filtrate plus concentrations of hormone *A* over the range of 0–6 units/ml. A second series, similar to the first but made with distilled water instead of ♂ filtrate, served as controls. Into 10-ml. samples of each of these solutions were placed two ♂ test plants which were washed in several changes of distilled water immediately before use. At the end of two hours counts of the antheridial branches were made. The results are given in table 2. A number of facts are apparent when these data are analyzed: (1) The filtrate from the vegetative ♂ plants contains an active substance, hormone *A'*. This sub-

TABLE 2

EFFECT OF ♂ FILTRATE ON THE REACTION OF ♂ PLANTS TO HORMONE *A*. REACTION INTENSITY IS EXPRESSED AS AVERAGE NUMBER OF ANTHERIDIAL BRANCHES

Hormone *A*, units/ml.	0	1	2	3	4	5	6
Control (no ♂ filtrate)	0	0.6	1.7	2.7	3.0	3.6	4.2
♂ filtrate added	0	11.8	12.6	13.1	12.4	10.4	12.5

stance (*a*) in the absence of hormone *A* is inactive in inducing the formation of antheridial branches, but (*b*) in the presence of hormone *A* activates this hormone or augments its activity. (2) The ♂ filtrate, in the presence of hormone *A* within the range of concentrations here used, determines the intensity of the ♂ reaction.

Comparison of the results in the foregoing experiments with those to follow will show that the amount of activation of ♂ filtrate varies appreciably among different samples of ♂ filtrate. Preliminary attempts to standardize the conditions of hormone *A'* production and to establish a quantitative test for its relative concentration have been unsuccessful. The experiments to follow, however, will show something of the activity of hormone *A'* and its relation to the rôle of hormone *A* in inducing the formation of antheridial branches on the ♂ plant.

The interaction of hormones *A* and *A'* is best shown in the following experiment. Two series of test solutions were prepared, and all the samples were tested simultaneously with carefully washed ♂ plants. In the first series hormone *A* in concentrations of 0.0, 0.006, 0.06, 0.6, and 6.0 units/ml. were added to 10-ml. samples of undiluted ♂ filtrate as prepared in the preceding experiment. The second series consisted of 10-ml. samples each containing 6.0 units/ml. hormone *A* plus 100, 50, 10 and 0% of ♂ filtrate. Average antheridial branch production (reaction intensity) for each solution was determined at the end of two hours. The results in figure 1 and table 3 clearly show that the number of antheridial branches produced by the ♂ plant varies in direct proportion to the concentration of both hormones *A* and *A'*, and that for a greater than minimal reaction both hormones must be present in adequate quantity. Thus a deficiency in either results in a low intensity reaction. The absolute necessity of

hormone A (from the ♀) for the production of antheridial branches on the ♂ plant is again demonstrated by these results.

TABLE 3

INTERACTION OF HORMONE A AND HORMONE A' IN THE PRODUCTION OF ANTHERIDIAL BRANCHES

Per cent conc. ♂ filtrate	←——100——→				50	10	0
Conc. hormone A, units/ml.	0.0	0.006	0.06	0.6	←——6.0——→		
Reaction intensity	0	2.9	6.7	11.0	27.5	20.6	11.5	7.5

The fact that the ♂ plant in the solution in which no ♂ filtrate was added still gave a reaction cannot be interpreted as meaning that the factor secreted by the ♂ plant is not equally essential. It will be recalled from the first two experiments reported above that the ♂ plants secrete sufficient quantity of hormone A' *during the two-hour testing period* to affect significantly the intensity of their reaction.

A marked rhythmic variation in the response of ♂ plants to a standard solution of hormone A has been described in an earlier paper.[2] No means have been found to change or eliminate this variation through rigid control of the conditions under which the plants are grown and tested. The substance secreted by the ♂ plant was naturally suspected of being responsible for this variation. Since hormone A' increases the activity of hormone A, is secreted by the ♂ plant and can limit the reaction within a critical concentration range, it seemed entirely possible that such variation resulted from the elaboration and secretion into the test fluid of varying amounts of the hormone at different times.

The results of a preliminary experiment seemed to justify this belief. At four-hour intervals carefully washed ♂ plants were placed in 10-ml.

TABLE 4

RELATION OF HORMONE A' TO THE RHYTHMIC VARIATION OF THE RESPONSE OF ♂ PLANT TO HORMONE A, EXPRESSED AS AVERAGE NUMBER OF ANTHERIDIAL BRANCHES

	TIME (NOV. 18–19, 1940)				
	10 A. M	2 P. M.	6 P. M.	10 P. M.	2 A. M.
Standard hormone A, 6 units/ml.	24.7	5.9	12.5	14.1	16.1
Standard hormone A + ♂ filtrate	24.9	21.7	..	21.8	21.7

samples of (1) standard hormone A solution (6 units/ml.) and (2) ♂ filtrate to which had been added hormone A in equal concentration. Counts of antheridial branches were made at the end of the four-hour intervals. These tests were performed under the conditions earlier described for the biological assaying of hormone A.[2, 4] The average number of antheridial branches produced in each of the solutions at four-hour intervals is given in table 4. In the series using only hormone A the coefficient of variation

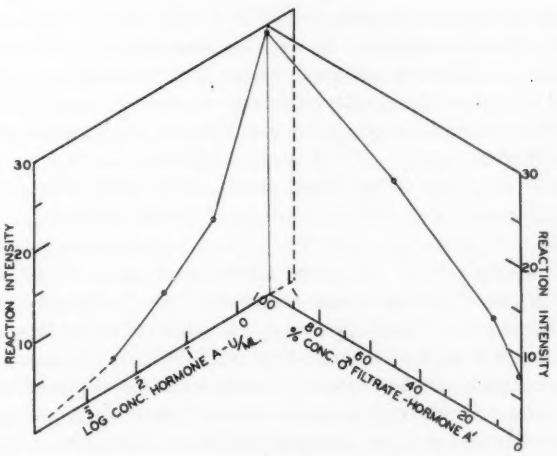

FIGURE 1

Interaction of hormones A and A' in the production of antheridial branches.

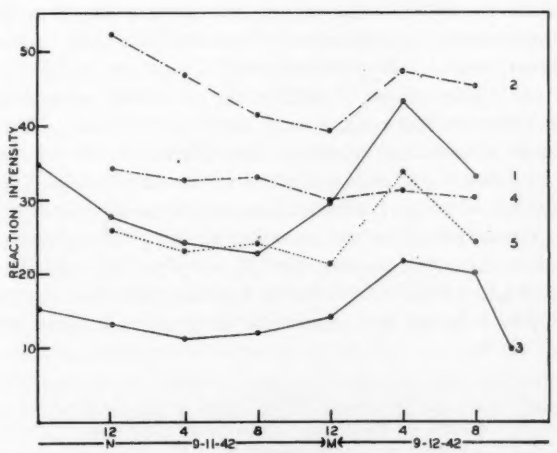

FIGURE 2

Effect of male filtrate on the variation in response of male plants to hormone A. The curves represent the reaction intensities of male plants in the following solutions: (1) 60 units of hormone A/ml. in distilled water, (2) 60 units hormone A/ml. in ♂ filtrate, (3) 6 units hormone A/ml. in distilled water, (4) 6 units hormone A/ml. in ♂ filtrate, and (5) 6 units hormone A/ml. in samples of ♂ filtrate collected at the times indicated.

in the reaction of the ♂ plants was 50.5 ± 4.25%. In the series containing added ♂ filtrate the coefficient of variation was reduced to 25.1 ± 2.1%.[5]

Later a more extensive and more adequately controlled experiment was conducted to retest the results obtained in the preliminary experiment above. Four solutions were made up as follows: (1) 60 units of hormone *A*/ml. in distilled water, (2) 60 units of hormone *A*/ml. in undiluted filtrate from ♂ plants as described above, (3) 6 units hormone *A*/ml. in distilled water and (4) 6 units of hormone *A*/ml. in ♂ filtrate. Two carefully washed ♂ plants were placed in a 10-ml. sample of each of these solutions, and at the end of four hours counts were made of the antheridial branches on the ♂ plants in each solution. This procedure was repeated at four-hour intervals during 24 hours. In addition to the four series thus established, a fifth series (5) was set up as follows: At the beginning of the experiment twenty ♂ plants were carefully washed and placed in 10 ml. of distilled water. At the end of each four-hour period the liquid was poured off these plants and replaced by fresh distilled water. The samples which were poured off were saved, and at the end of the 24-hour testing period hormone *A* was added to each in a concentration of 6 units/ml. The six samples, comprising series 5, were then tested with ♂ plants simultaneously with a standard (6 units/ml.) hormone *A* solution as control. The results of these five series are plotted in figure 2. It is obvious on examination of these data that under these conditions the variation of ♂ response is not eliminated by the addition of hormone *A'*, but in each of the series containing added ♂ filtrate the variation is considerably less than in the corresponding control series lacking ♂ filtrate. The mean and the coefficient of variation in each of the different series are presented in table 5. In series 2, containing ♂ filtrate, the amplitude of the variation is only half that of series 1, which contains the same concentration of hormone *A*. The amplitude of the variation in series 4 is approximately one-fourth that in the corresponding control series lacking added ♂ filtrate. These results taken with those in table 3 demonstrate that the variation of the ♂ response is due at least in part to the quantity of some active factor

TABLE 5

HORMONE *A'* CONTROL OF RHYTHMIC VARIATION IN RESPONSE OF ♂ PLANTS TO HORMONE *A*

SERIES	MEAN	COEFF. OF VARIATION—%
1. 60 units/ml. *A*	30.2±0.81	31.6±2.08
2. 60 units/ml. *A* + ♂ filtrate	45.6±0.76	18.2±1.21
3. 6 units/ml. *A*	14.9±0.52	42.6±3.42
4. 6 units/ml. *A* + ♂ filtrate	31.9±0.47	16.5±1.11
5. 6 units/ml. *A* + filtrate from 20 ♂ plants at 4-hour intervals	25.4±0.72	30.5±2.28

secreted by the ♂ mycelium. The rôle of the ♂ secretion is even more strikingly shown in the results of series 5. The curve of the intensity of the reactions induced in the four-hour samples of ♂ filtrate follows rather closely the curves of series 1 and 3, to which no ♂ filtrate was added. The agreement in shape among these curves demonstrates conclusively that the ♂ plant secretes at different times varying quantities of hormone A', and that this variation in amount of secretion results in the rhythmic variation in the response of the plants to a constant concentration of hormone A.

Nothing is known as yet about the nature of the mechanism of hormone A' effect. Elucidation of the nature of this effect will be difficult, since the only means of detecting the presence and activity of the substance lies in the reaction of the plant which secretes it. Nor is it possible at the present time to say whether or not the substance is indispensable for the formation of antheridial hyphae. It seems entirely possible that it is essential, since carefully washed ♂ plants give a reaction of very low intensity at the time of low hormone A' productivity. Before these problems can be solved some means must be found either to inhibit the production of A' or to block its effect. Working out a method for the quantitative determination of the active substance will be similarly handicapped. On the other hand, it is difficult to explain the tremendous variation in response of ♂ plants to hormone A on the basis of quantitative differences of a single substance, and it is entirely possible that more than a single active factor is secreted into the medium by the ♂ plant.

Summary.—In addition to the four specific substances earlier described as initiating and coördinating the sexual reaction in two heterothallic species of *Achlya*, a fifth active factor, hormone A', secreted by the vegetative ♂ mycelium, has now been demonstrated. The activity of this substance has been shown in the following ways: (1) the response of the ♂ plant to hormone A increases (*a*) with successively introduced ♂ plants into a single sample of hormone A solution, (*b*) with the number of ♂ plants per unit volume of hormone A solution, (*c*) in the presence of ♂ filtrate and (*d*) with increasing concentration of ♂ filtrate; and (2) the variation in the response of ♂ plants to hormone A depends upon the rhythmic variation in the quantity of hormone A' produced by the ♂ plant.

* The early portions of this work were conducted in the Kerckhoff Biological Laboratories of the California Institute of Technology, Pasadena, Calif., during the tenure of a National Research Fellowship in Botany.

Report of work carried out with the aid of Works Progress Administration, official project No. 165-1-07-172, Work Project No. N-12165. The author wishes to express his indebtedness to Mr. Harvey W. Tomlin for his invaluable assistance.

[1] Raper, John R., *Science*, **89**, 321 (1939); *Amer. Jour. Bot.*, **26**, 639 (1939); *Ibid.*, **27**, 162 (1940).

[2] Raper, John R., *Amer. Jour. Bot.*, **29**, 159 (1942).

[3] Hormone *A* from *Achlya bisexualis* ♀ has been used in these studies for convenience and in order to obtain as nearly reproducible solutions as possible. Tests have been made with hormone *A* from *Achlya ambisexualis* ♀ with entirely comparable results.

[4] Raper, John R., and Haagen-Smit, A. J., *Jour. Biol. Chem.*, **143**, 311 (1942).

[5] In this and the following experiment the coefficient of variation has been calculated from the total number of individual counts rather than from the averages given in table 4 and figure 2. Each of these values represents the average number of antheridial branches on twenty hyphae chosen at random.

OXIDIZED COTTON, AN IMMUNOLOGICALLY SPECIFIC POLYSACCHARIDE

BY MICHAEL HEIDELBERGER AND GLADYS L. HOBBY

PRESBYTERIAN HOSPITAL, COLUMBIA UNIVERSITY

Communicated October 19, 1942

The immunologically specific polysaccharide of Type III pneumococcus, to which the type-specificity and virulence of this pathogenic microörganism are due,[1] is made up of a chain of aldobionic acid,[2] more particularly, cellobiuronic acid,[3] units. The corresponding capsular carbohydrate of the Type VIII pneumococcus contains the same aldobionic acid unit,[4] and in addition, roughly two glucose molecules for every such unit.[5] The close chemical relationship of these polysaccharides is reflected in their serological behavior: each polysaccharide not only precipitates antisera to the homologous pneumococcus type, but the Type III polysaccharide also precipitates a portion of the antibodies in Type VIII antiserum and the Type VIII polysaccharide throws down a portion of the antibodies in Type III antiserum.[6] Evidence has been given that this cross-reactivity is due to some multiple of the repeating unit containing cellobiuronic acid in common.[5]

It has recently been shown that cotton may be oxidized by means of nitrogen dioxide to products containing varying percentages of carboxyl.[7] These products would therefore contain cellobiuronic acid units, separated by glucose, at intervals in the long cellulose chain and might not unreasonably be expected to react specifically with Type VIII antipneumococcus horse serum, or even with the Type III antiserum. Samples of the oxidized cotton containing 16 and 21 per cent of $-CH_2OH$ oxidized to $-COOH$ were kindly supplied by Drs. Yackel, Unruh and Kenyon of the Eastman Kodak Laboratories. While it was possible to fractionate the material to some extent, it was most convenient to dissolve weighed quantities in excess *N* $NaHCO_3$ solution, a slow process in the case of the sample of lower $-COOH$ content, neutralize the excess of bicarbonate and dilute, first with

water to approximately 0.15 *N* Na^+ content and then with 0.15 *N*, or physiological, NaCl solution. Qualitative tests showed that both samples, at high dilutions, precipitated antipneumococcus Types III and VIII horse sera, and failed to react with Type I and Type II antisera. Quantitative data[8] for solutions of the sample with 16% —COOH indicated a degree of cross-reactivity entirely comparable to that of the corresponding pneumococcus specific polysaccharides.[5, 9] Moreover, the antibody fraction reactive with the oxidized cotton was mainly the portion cross-reactive with the pneumococcus polysaccharide of heterologous type.

Analyses, calculated to 1.0 ml. portions, with Type VIII antipneumococcus horse serum 909, containing 1.0 mg. of Type VIII anticarbohydrate nitrogen per ml.:[10]

POLYSACCHARIDE ADDED	QUANTITY OF POLYSACCHARIDE, MG.	ANTIBODY NITROGEN PRECIPITATED, MG.
Type III pneumococcus	0.13	0.202
Type III pneumococcus	0.4	0.278
Oxidized cotton (16% —COOH)	0.02	0.158
Oxidized cotton (16% —COOH)	0.07	0.273

The supernatant from the last tube, plus 0.2 mg. Type III polysaccharide yielded 0.075 mg. of cross-reactive antibody N, showing that roughly two-thirds of the cross-reactive antibody had been removed by the cotton solution.

Analyses, calculated to 1.0-ml. portions, with Type III antipneumococcus horse serum 792, containing 0.7 mg. of Type III anticarbohydrate nitrogen per ml.[10]

POLYSACCHARIDE ADDED	QUANTITY OF POLYSACCHARIDE, MG.	ANTIBODY NITROGEN PRECIPITATED, MG.
Type VIII pneumococcus	0.02	0.111
Type VIII pneumococcus	0.2	0.190
Oxidized cotton	0.008	0.080
Oxidized cotton	0.05	0.165
Oxidized cotton	0.2	0.150

The supernatant from the last tube, plus 0.1 mg. Type VIII specific polysaccharide, yielded only 0.011 mg. of antibody N, showing that most of the cross-reactive antibody had been removed by the cotton solution. The supernatant from the next-to-last tube gave 0.016 mg. antibody N with Type VIII polysaccharide, while that from the 0.008-cotton tube gave 0.065 mg. The sums of the two sets of values are nearly equal to the antibody N precipitated by the Type VIII polysaccharide alone.

As would be expected from its chemical structure, the oxidized cotton with 16% —COOH corresponds more closely in its immunological behavior to the Type VIII substance than to the Type III polysaccharide.

Attempts were also made to confer active protection upon mice with doses of oxidized cotton ranging from 0.005 mg. down to 0.00001 mg.,[11] but the material failed to protect against 10 to 100 minimal lethal doses of an extremely virulent strain of Type VIII pneumococcus. However, the same amounts of the Type III and Type VIII specific polysaccharides also failed to give protection against this strain.

The precipitation data with the oxidized cotton again emphasize the strict correlation between chemical constitution and immunological specificity and show that predictions as to reactivity may be made when the constitution of the repeating unit responsible for that reactivity is known.

[1] Heidelberger, M., and Avery, O. T., *Jour. Exp. Med.*, **38**, 73 (1923); **40**, 301 (1924). Avery, O. T., and Heidelberger, M., *Ibid.*, **42**, 367 (1925).

[2] Heidelberger, M., and Goebel, W. F., *Jour. Biol. Chem.*, **70**, 613 (1926); **74**, 613 (1927).

[3] Hotchkiss, R. D., and Goebel, W. F., *Ibid.*, **121**, 195 (1937). Reeves, R. E., and Goebel, W. F., *Ibid.*, **139**, 511 (1941). Adams, M. H., Reeves, R. E., and Goebel, W. F., *Ibid.*, **140**, 181 (1941).

[4] Goebel, W. F., *Ibid.*, **110**, 391 (1935).

[5] Heidelberger, M., Kabat, E. A., and Shrivastava, D. L., *Jour. Exp. Med.*, **65**, 487 (1937).

[6] Sugg, J. Y., Gaspari, E. L., Fleming, W. L., and Neill, J. M., *Ibid.*, **47**, 917 (1928). Cooper, G., Edwards, M., and Rosenstein, C., *Ibid.*, **49**, 461 (1929). Also Ref. 5. Antipneumococcus Types III and VIII horse sera show this cross-precipitation more regularly than do the corresponding rabbit sera.

[7] Yackel, E. C., and Kenyon, W. O., *Jour. Am. Chem. Soc.*, **64**, 121 (1942). Unruh, C. C., and Kenyon, W. O., *Ibid.*, **64**, 127 (1942).

[8] Heidelberger, M., Kendall, F. E., and Soo Hoo, C. M., *Jour. Exp. Med.*, **58**, 137 (1933). Heidelberger, M., and Kendall, F. E., *Ibid.*, **61**, 559; **62**, 697 (1935).

[9] Heidelberger, M., Kabat, E. A., and Mayer, M., *Ibid.*, **75**, 35 (1942).

[10] After removal of antibodies precipitated by pneumococcus C-substance.

[11] Schiemann, O., and Casper, W., *Zeitschr. f. Hyg.*, **108**, 220 (1927). Felton, L. D., *Jour. Infect. Dis.*, **56**, 101 (1935).

---

*THE GENETIC NATURE OF X-RAY INDUCED CHANGES IN POLLEN**

By Charles M. Rick

Division of Truck Crops, University of California

Communicated October 28, 1942

Ample evidence exists in genetic literature to suggest that many more pollen characters are genetically self-determined in the pollen than have already been reported. Some of these cases—for example, the genes determining small pollen in *Zea*—were discovered by virtue of the fact that

small pollen is less viable and hence causes distorted ratios of more conspicuous characters to which the genes for small pollen size are linked.[1] Some other respects in which genetic autonomy occurs in pollen include: chemical composition,[2] viability in the many instances of semi-sterility and behavior as in the many examples of oppositional self-sterility allelomorphs.

With the known mutation inducing agents available, it should be possible to demonstrate whether or not certain characters are determined in this manner. Since size is usually governed by a large number of genetic factors, each of relatively small effect and each integrated with others in its action, mutation in a single gene of this type in pollen would be very difficult to detect. Mutation of a large number, however, in pollen where the haploid condition would permit expression of recessive as well as dominant changes, should be readily detectable as an increase in variation of size. Accordingly, experiments were performed to detect whether or not x-rays would induce any genetic response in size of pollen. Lethal effects, i.e., changes resulting in abortion of pollen, were also studied.

Genetically self-determined pollen abortion, whether conditioned by chromosomal deficiencies and duplications or by gene mutation, has been observed frequently. There are also reports in the literature of a similar control of microspore and pollen size.[3] Nevertheless, pollen size may also be determined by the sporophyte producing the pollen instead of by the gametophyte itself.[4]

Typical of pollen measurements in general, the intra-treatment variability of sample values was much greater than would be expected on the basis of pure chance variation. In order to cope with this natural fluctuation in comparing different treatments statistically, many samples of small size were taken and n was taken as the number of samples, instead of the total number of grains observed, as the basis for comparisons between treatments. Pollen abortion, mean length of grain and variance of those lengths were calculated for each sample. Means and standard errors were then calculated for the distribution of these sample values. In the following report a random sample of 25 was adopted for measurement of lengths, and 100 for frequency of pollen abortion.

The pollen collections were examined in aceto-carmine in order to estimate pollen abortion in the same collection used for measurement of lengths. Examination was made immediately after mounting. A pollen grain was considered aborted if it was devoid of cytoplasmic contents as indicated by the aceto-carmine stain. Length of grain is the most satisfactory dimension for measurement of the elongate grains of *Tradescantia* and *Pisum* lines used here. Lengths were measured by means of an ocular micrometer.

Diploid-Tetraploid Comparisons.—A comparison between diploid and closely related autotetraploid lines should offer a test of the hypothesis of

x-ray induced mutations in pollen grains. Stadler[5] has demonstrated conclusively that in the cereals the rate of x-ray induced mutation drops rapidly with increasing degree of polyploidy. The pollen produced by autotetraploids is diploid, consisting of two identical sets of chromosomes. Since every gene exists in duplicate, a mutation from the dominant to recessive condition of any gene will gain no phenotypic expression because of the presence of the dominant allele except in the very rare event that the same recessive mutation is induced in both alleles. On the other hand, the haploid pollen produced by diploids should show phenotypic expression of recessive mutations. Since the great majority of x-ray induced mutations are recessive, this comparison should be illuminating.

In the x-ray treatments clones of *Tradescantia* species were used because of the data available on the timing of the cycle of development of the male gametophyte.[6] Collections were made from the eight to eleventh day after the material had been irradiated. In terms of the developmental cycle these collections were taken from buds in which the pollen mother cells had just completed meiosis and the microspores were experiencing the first half of the post-meiotic resting stage at the time of treatment. The conditions of radiation are given in table 1.

TABLE 1

COMPARISON OF X-RAY EFFECTS ON ABORTION OF POLLEN, MEAN AND VARIANCE OF LENGTH OF GRAIN IN DIPLOID AND AUTOTETRAPLOID *Tradescantia* SPECIES

200 kv., 15 ma., 81 cm., $^1/_2$ mm. copper

SPECIES	TREATMENT	NUMBER OF SAMPLES	POLLEN ABORTION, %	LENGTH OF POLLEN GRAIN MEAN, μ	LENGTH OF POLLEN GRAIN VARIANCE, μ^2
T. paludosa					
Diploid	Control	16	4.99 ± 0.49	49.11 ± 0.19	0.742 ± 0.042
	400 r	18	10.98 ± 0.83	47.89 ± 0.24	2.163 ± 0.158
T. canaliculata					
Autotetraploid	Control	20	13.10 ± 0.90	57.24 ± 0.27	1.322 ± 0.062
	400 r	18	13.20 ± 0.79	56.34 ± 0.45	1.308 ± 0.138
Clone No. 374*					
Diploid	Control	10	22.2 ± 1.2	49.38 ± 0.26	2.549 ± 0.037
	200 r	10	28.87 ± 1.43	47.99 ± 0.14	3.043 ± 0.149
T. virginiana					
Autotetraploid	Control	19	26.03 ± 0.92	56.38 ± 0.33	1.879 ± 0.115
	400 r	19	26.84 ± 1.25	59.13 ± 0.20	1.940 ± 0.088

* A segregate from the cross, *T. canaliculata* x *T. humilis*.

Two clones of diploid and two of tetraploid *Tradescantia* species were used in this study. Evidence of the autotetraploid nature of the tetraploid species, *T. virginiana* L. and *T. canaliculata* Raf., has been presented by

Anderson and Sax.[7] Of the diploid clones used, one was identified as *T. paludosa* Anderson and Woodson and the other was a segregate from the cross, *T. canaliculata* Raf. x *T. humilis* Rose. The diploid *T. paludosa* and the tetraploid *T. canaliculata* were irradiated simultaneously and grown under the same conditions. The other clones were treated at different times.

The results are summarized in table 1. In the diploid clones the radiation induced a very significant increase in pollen abortion and in variability of length and a decrease in mean length. These changes showed a slight gradual increase during the four-day period of collections, but in comparison with measurements of collections on preceding days, these values are at a new high level and for purposes of comparison can be safely considered as a unit. This pattern of x-ray effect is typical of many that have been investigated here in diploid *Tradescantia*.

This response to x-ray treatment of the haploid microspores would have occurred if the radiation induced mutations (chromosomal aberrations as well as gene mutations in the strict sense) which were immediately expressed in the size and viability of the microspores. The great preponderance of x-ray induced mutations reported in the literature has been of the negative type; therefore, it is no surprise that the changes encountered here have been mostly, if not entirely, in the direction of smaller grains, and that the percentage of aborted grains showed a significant increase. The changes involved will be described in more detail in a later publication.

The effect on the pollen of autotetraploid clones is strikingly different. The rate of pollen abortion and the variance of pollen grain length remain very little affected by radiation which caused quite significant changes in the diploid, thus bearing out the genetic interpretation. Mean length of grain seemed to be affected, in one instance to a nearly significantly lower level and in the other to a very significantly higher level. No explanation would seem to account for these peculiar trends. Of the three values, mean length in general shows the greatest fluctuations and lengths from the same plant may be subject to substantial increases and decreases over a period of time. Since the changes here are both positive and negative, it is doubtful whether they are related to the x-ray treatment.

In a comparison of the x-ray sensitivity of microspore chromosomes, Sax and Swanson[8] found the rate of aberration in chromosomes of diploid *Tradescantia* microspores to be only half that of the haploid. These results were expressed on a chromosome basis; if expressed as frequencies per cell, the sensitivity of haploid and diploid would be about equal. Yet, even if this might indicate a gene sensitivity reduced to one-half in the diploid microspore, the differences in table 1 would remain relatively unaffected. For instance, the negative changes in the autotetraploid clones

must be increased by factors of 3 to 25 before the differences reach the five per cent level of significance.

TABLE 2

COMPARATIVE EFFECTS OF X-RAYS APPLIED AT 3°C. AND 33°C. ON MEASUREMENTS OF POLLEN GRAINS

134 kv., 10 ma., 50 cm., no filters

Ten samples per treatment

TREATMENT NUMBER	RADIATION	TEMPERATURE, °C.	POLLEN ABORTION, %	LENGTH OF POLLEN	
				MEAN, μ	STANDARD DEVIATION, μ
20*	150 r	3	11.5 ± 0.9	47.11 ± 0.63	2.758 ± 0.096
21*	150 r	33	10.2 ± 0.9	47.67 ± 0.54	2.492 ± 0.082
24	None	3	9.5 ± 0.6	46.57 ± 0.59	1.836 ± 0.099
19	None	33	.....	46.75 ± 0.48	2.027 ± 0.082

* Lots 20 and 21 were irradiated simultaneously.

Effect of Temperature on X-ray Induced Changes.—Only one trial has been made of the effect of temperature. The pollen test of mutation rate offers advantages of rapidity and ease of manipulation for tests of this sort and might prove useful in future experiments where the contributory effects of dosage, intensity, temperature and other factors are studied. Inflorescences of a diploid clone of *Tradescantia paludosa* were irradiated simultaneously in cardboard containers of warm and cold water. The temperature differences were maintained during the period of treatment and for several hours afterward. The results and other conditions of the experiment are given in table 2. Temperature influenced the values significantly only in the case of variation of length and here the P value of the difference is 0.036, but the effect of radiation at low temperature is consistently more intense in each of the three measures. From the standpoint of direction and consistency alone these results are well within the realm of chance occurrences and larger samples will be needed from material treated with heavier doses before it can be said with certainty that low temperature enhances the x-ray effect in all respects.

The increased variability induced at 3° over 33°C. would be rather difficult to account for in terms of a physiological response other than a genetic one. The genetic interpretation is admittedly far from satisfactory, yet the point of interest here is that this response in variability of pollen size agrees with the response found in most other experiments, the genetic nature of which is undisputed. For instance, the frequency of x-ray induced chromosomal aberrations in *Tradescantia*[9] and in *Drosophila*[10] is greater when treatments were applied at lower temperatures. A similar response has been found in lethal mutations in *Drosophila*[11] and in chlorophyll deficient mutants in *Hordeum*[12] but these reports are contradicted

by the finding of no temperature effect by others.[13] The picture conveyed by the literature then is confused but there is agreement to the effect that there is no positive temperature coefficient of x-ray induced mutation and in this respect the response in pollen resembles mutation.

Inheritance of Pollen Size in Pisum sativum L.—As mentioned above there is ample evidence in the literature of genes governing size, viability and physiological activity of pollen, which segregate at meiosis and gain expression immediately in the grains to which they are contributed. In these cases the segregating types have been sharply distinguished and only one or a few gene pairs have been concerned in their determination. Although it did not seem unreasonable to suppose that pollen size could be regulated by multiple factors, it seemed highly desirable to have some such example in untreated material to compare with the x-ray treatments.

The naturally self-pollinated legumes offer a source of lines which are homozygous yet do not suffer the usual effects of inbreeding. In the garden pea, *Pisum sativum* L., varieties were found which differed in length of pollen grain. Crosses were made between large and small types and the F_1 hybrids were grown simultaneously with selfed progeny from the plants used as parents. For comparative data, samples were taken from the selfed progenies of the parents rather than from the parent plants themselves in order that the parent lines be grown under the same conditions as the hybrids. The homozygous condition of the parent plants attested by the uniformity of their progeny justified this measure.

TABLE 3

MEASUREMENTS OF POLLEN IN F_1 AND PARENTAL LINES OF GARDEN PEA

NUMBER OF LINE	DESCRIPTION	NUMBER OF SAMPLES	POLLEN ABORTION, %	LENGTH OF POLLEN GRAIN MEAN, μ	LENGTH OF POLLEN GRAIN VARIANCE, μ^2
42 *Ps* 8	F_1, 3 × 13	18	2.68 ± 0.55	50.68 ± 0.32	1.267 ± 0.105
42 *Ps* 18	F_1, 25 × 26	9	2.49 ± 0.72	52.13 ± 0.58	1.860 ± 0.271
42 *Ps* 22	F_1, 11 × 25	19	3.71 ± 0.44	51.25 ± 0.46	1.511 ± 0.075
42 *Ps* 28	F_1, 31 × 25	11	1.94 ± 0.46	51.05 ± 0.34	1.270 ± 0.070
42 *Ps* 3	P_1	19	2.42 ± 0.35	55.58 ± 0.21	0.602 ± 0.038
42 *Ps* 11	P_1	18	2.89 ± 0.30	54.39 ± 0.29	0.590 ± 0.040
42 *Ps* 13	P_1	18	5.91 ± 0.72	49.67 ± 0.37	0.545 ± 0.043
42 *Ps* 25	P_1	25	5.27 ± 0.44	49.57 ± 0.32	0.685 ± 0.045
42 *Ps* 26	P_1	20	3.17 ± 0.39	54.01 ± 0.14	0.535 ± 0.023
42 *Ps* 31	P_1	3	10 ±	53.76 ± 0.42	0.552 ± 0.050

Ample data for statistical comparisons are available from four different F_1's and their parents. The observations are presented in table 3 and figure 1. In every case pollen of the hybrid was intermediate in size between its two parents; furthermore, the variation of the hybrid pollen grain length significantly exceeded that of each parent in every case. In the case of

least difference, i.e., the F_1, 42 *Ps* 18 and its parent, 42 *Ps* 25, the t value of the difference is 2.63 with a corresponding P value of less than 0.01. This is precisely the effect expected if size of pollen were to some extent governed by the genotype of the pollen itself and if a large number of genes, each having a small effect, were involved. A possible source of internal instability might be hybridity for chromosome rearrangement, say inversions or translocations. Sufficient buds of the hybrids were not available to permit a cytological study to detect such hybridity but the four pea hybrids show no increased percentage of pollen abortion which is characteristic of inversion and translocation heterozygotes.

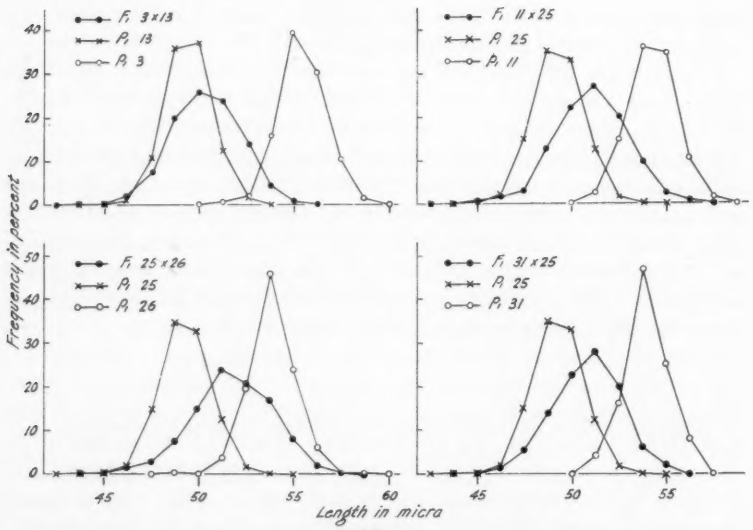

FIGURE 1

Frequency distributions of length of pollen in F_1 and parental lines of garden pea.

Unless a very great number of factors interact to determine size differences in these hybrids, the distribution of lengths in the hybrids might be expected to completely cover the range of either parent. Also, the presence of only a few determiners of small size in the larger parent or vice versa might lead to considerable transgressive variation. Since these F_1 distributions are all contained within the range of either parent and one in particular, 42 *Ps* 8, fails to cover the entire range of its parents, it is possible that even in these hybrids size is determined maternally to a limited extent. Nevertheless, the consistent tendency of all hybrids to be more variable than either parent certainly points to a multifactorial genetic determination of size within each grain.

Summary.—X-rays applied to microspores of diploid species of *Tradescantia* shortly after meiosis cause a significant increase in variability of length of the subsequently developed pollen grains and in the percentage of aborted pollen. There is also a significant decrease in the mean lengths of grains. Similarly treated autotetraploid species of *Tradescantia* show no significant changes. This difference is interpreted to indicate that in the diploid pollen of autotetraploids any recessive mutation is masked by its dominant allele.

X-rays induced significantly greater variability of pollen length when applied to diploid *Tradescantia* at 3° than when applied at 33°C.

In consideration of these results it is concluded that size and viability of pollen are, at least in part, genetically self-determined and that the changes observed are the consequence of mutations induced by the x-ray treatment.

These conclusions are supported by an analysis of genetic variation in pollen length in *Pisum sativum*.

* The writer takes great pleasure in acknowledging the many constructive suggestions offered by Dr. Karl Sax of Harvard University. Martha O. Rick, the writer's wife, aided to a great extent in many phases of the work, and to her the writer wishes to express his sincerest obligations. The writer is indebted to Dr. P. C. Aebersold of the Crocker Radiation Laboratory for his kind help in applying the x-rays.

[1] Mangelsdorf, P. C., *Jour. Hered.*, **23**, 289–295 (1932); Rhoades, M. M., and Rhoades, Virginia H., *Genetics*, **24**, 302–314 (1939).

[2] Brink, R. A., and MacGillivray, J. H., *Amer. Jour. Bot.*, **11**, 465–469 (1924); Demerec, M., *Ibid.*, **11**, 461–464 (1924); Parnell, F. R., *Jour. Genet.*, **11**, 209–212 (1921).

[3] Mangelsdorf, P. C., loc. cit.; Renner, O., *Zeit. Bot.*, **11**, 305–380 (1919); Satina, Sophia, and Blakeslee, A. F., *Amer. Jour. Bot.*, **24**, 518–527 (1937); Sax, K., *Genetics*, **22**, 523–533 (1937); Sinotô, Y., *Cytologia*, **1**, 109–191 (1929).

[4] Bateson, W., Saunders, E. R., Punnett, R. C., and Hurst, C. C., *Rept. Evol. Com. Roy. Soc. Rept.*, **2**, 154 pp. (1905). Another example of sporophytic control is the difference in size and color of pollen maintained in anthers of different levels in the flower of *Lythrum Salicaria* L.

[5] Stadler, L. J., these PROCEEDINGS, **15**, 876–881 (1929).

[6] Sax, K., *Genetics*, **23**, 494–516 (1938).

[7] Anderson, E., and Sax, K., *Bot. Gaz.*, **97**, 433–476 (1936).

[8] Sax, K., and Swanson, C. P., *Amer. Jour. Bot.*, **28**, 52–59 (1941).

[9] Fabergé, A. C., *Jour. Genet.*, **39**, 229–248 (1940); Rick, C. M., *Genetics*, **25**, 466–482 (1940); Sax, K., and Enzmann, E. V., these PROCEEDINGS, **25**, 397–405 (1939).

[10] Mickey, G. H., *Genetica*, **21**, 386–407 (1939).

[11] Medvedev, N. N., *Compt. Rend. Acad. Sci. U.R.S.S.*, **4**, 283–285 (1935); Medvedev, N. N., *Ibid.*, **19**, 301–302 (1938); Timoféeff-Ressovsky, N. W., *Mutationsforschung in der Vererbungslehre*, 184 pp., Theodor Steinkopf, Dresden (1937).

[12] Rick, C. M. (unpublished).

[13] Stadler, L. J., *Jour. Hered.*, **21**, 3–19 (1930); Stadler, L. J., *Sci. Agr.*, **11**, 645–661 (1931); Timoféeff-Ressovsky, N. W., und Zimmer, K. G., *Biol. Zbl.*, **59**, 358–362 (1939).

THE LOWER HYDRATES OF SOAP

M. J. BUERGER, L. B. SMITH, A. DE BRETTEVILLE, JR., AND F. V. RYER

Communicated October 24, 1942

1. Introduction.—It has been pointed out[1] that the non-recognition of the importance of minor "impurities" may lead to grave misinterpretations of the stability relations between crystalline phases. Conversely, anomalous phase relationships are cause for suspecting that "impurities" may be playing rôles which cannot be neglected. Some of the relationships supposed to exist between certain forms of soap, for example, are almost incredible, and strongly suggest that the soap forms in question do not fall into a simple one-component system, as hitherto supposed, but are parts rather of a two-component system. Our experimental investigation of the relationship between some of these soap forms herein reported indeed proves that the second component is water, and that not a few of the all-too-numerous and supposedly anhydrous soap forms are actually hydrates. This paper is devoted to a discussion of the relation of the lower hydrates and anhydrous soap to one another, together with a brief résumé of the experimental evidence for these relations.

2. Extension of the Temperature Range of the So-Called α Form.—According to Thiessen and Stauff,[2] the form of sodium stearate which precipitates from alcoholic solutions, called by them the α form, is metastable and, on heating, changes irreversibly to the stable β form at about 52°C. The further relationships between these forms as given by them are most anomalous. These relationships would be understandable if the two forms were not the same compound. This suspicion is strengthened by the following experiment: Since the α form transforms to the β form at about 52°C., it should be possible to grow the latter from an alcoholic solution at temperatures greater than 52°C. Accordingly, we tried to precipitate crystals of the "stable" β form from boiling alcohol at 78°C. X-ray photographs proved the precipitate to be the "metastable" α form, which according to the transformation experiments of Thiessen and Stauff ought not to have been preserved at temperatures above 52°C. The conclusion is inescapable that something in the alcohol is vital to the formation of the α form, since, if it is exposed to this temperature out of the presence of alcohol, it immediately transforms to the β form. We strongly suspected on several grounds that the two forms differ in water content and that the water is supplied by the alcohol solvent (it is almost impossible to keep even absolute alcohol water-free during the course of an experiment).

3. Transformation from the α to the β Form and Its Reversal.—If the two forms of sodium stearate differ by water content, then it is evident that there is a possibility of passing from either form to the other by the ap-

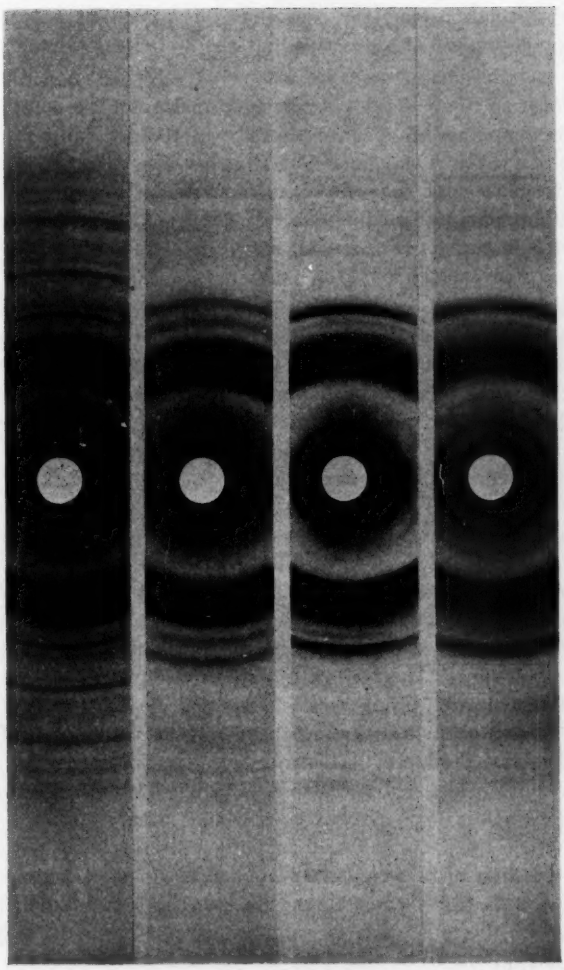

FIGURE 1

X-ray powder photographs (θ range, 0 to 43°) of the following forms of hydrous and anhydrous sodium stearate, left to right: NaSt·$^1/_2H_2O$ ("α NaSt"); NaSt·$^1/_8H_2O$ ("β NaSt"); NaSt ("γ NaSt"); form assumed by NaSt after having been heated to any temperature above about 117°C.

propriate addition or removal of water. According to our experiments, the removal of water by any of the following means is indeed accompanied

by a transformation from the α form to the β form, as proved by x-ray powder photographs of the original material and of the product:

(a) Heating to about 52°C.; a transformation of the α form to the β form is accompanied by an evolution of water which can be detected on the walls of a tube. This can be done in either an open or closed tube.

(b) Evacuation of the α form at room temperature (especially easy with sodium palmitate).

(c) Grinding at room temperature.

The above evidence might possibly be interpreted as various actions triggering off an inversion from a monotropic to stable form. The following reversal of the transformation cannot be so interpreted, however: If the supposedly stable β form is heated in a sealed tube with a drop of water at 47°C., the supposedly metastable modification forms at the expense of the supposedly stable one. The only interpretation of this experiment is that a compound of greater hydration is called forth by the presence of the added water.

4. *Transformations between the β Form and γ Form.*—When the β form of sodium stearate is heated, it transforms at about 103°C. to a form called the γ form by de Bretteville.[3, 4] It previously had been assumed that this transformation is a polymorphic inversion. This relation came under suspicion when we noticed that it was prevented by carrying out the heating in a sealed-off tube. This lead us to suspect that the β form was hydrous and that dehydration is prevented in a small closed space. To confirm this, we carried out a transformation cycle with sodium palmitate. The β form was first produced. This was heated in an open tube at 106°C. The x-ray pattern then showed it to have changed into a different form (de Bretteville's γ form). This was then heated with water in a sealed-off tube at either 100°C. or 200°C. In both cases the x-ray pattern of the product was that of the original β form. Subsequent heating in the open at about 106°C. transformed this again into the γ form. This cycle was repeated with the same material as many times as desired. It plainly indicates that water can be added to the γ form to produce the β form and that water can be driven off from the β form to reduce it to the γ form. Thus, the β form is a hydrous form.

5. *Composition of the Several Soap Forms.*—The several pieces of evidence just given establish the fact that the several forms of soap are hydrous in the following order:

α form (most hydrated)
β form
γ form (least hydrated)

The only remaining question is what is the absolute degree of hydration of each form. Xylol tests established the following hydrate compositions:

α form—sodium stearate $^1/_2$ H_2O
β form—sodium stearate $^1/_8$ H_2O
γ form—sodium stearate

6. X-ray Photographs of Soap Forms.—In figure 1 we reproduce x-ray powder photographs of the less hydrated forms of soap. In an accompanying paper,[5] x-ray crystallographic data for the hemihydrates are given.

7. Other Soap Hydrates.—We have also investigated other hydrates and hope to publish data for them at a later time.

[1] Buerger, M. J., "The General Rôle of Composition in Polymorphism," *Proc. Nat. Acad. Sci.*, **22** 685–689 (1936).

[2] Thiessen, Peter A, and Stauff, Joachim, "Feinbau und Umwandlungen kristallisierter Alkalisalze langkettiger Fettsäuren," *Zeit. Physikal. Chem. (A)*, **176** 397–429 (1936).

[3] Bretteville, A. de, Jr., "X-ray Diffraction Study of Oriented Soaps," Thesis, Stanford University (1942).

[4] Bretteville, Alexander de, Jr., and McBain, J. W., "X-ray Evidence for a Third Polymorphic Form of Sodium Stearate," *Science*, **96**, 470–471 (1942).

[5] Buerger, M. J., "The Characteristics of Soap Hemihydrate Crystals," *Proc. Nat. Acad. Sci.*, **28** 529–535 (1942).

THE CHARACTERISTICS OF SOAP HEMIHYDRATE CRYSTALS

BY M. J. BUERGER

MINERALOGICAL LABORATORY, MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Communicated October 24, 1942

1. Introduction.—In an accompanying paper[1] it is shown that Thiessen and Stauff's[2] α sodium stearate and α sodium palmitate are actually the hemihydrates of these compounds. Crystallographic observations on these were also published by Thiessen and Stauff, but their conclusions are in error because the rotating-crystal and powder x-ray diffraction methods, which they employed, are impotent when used to study crystals of large cells and low symmetry. In this paper, the results of a restudy of these crystals by more powerful methods are given.

While a more specific transformation from Thiessen and Stauff's axes is given beyond, the following orientation changes will be of value to those familiar with the literature of soap crystals in understanding the notation used in this paper:

BUERGER'S AXES	AXES CORRESPONDING WITH THIESSEN AND STAUFF'S LATTICE CONSTANTS	AXES CORRESPONDING WITH THIESSEN AND STAUFF'S FIGURE 1
a	b	a
b	a	b
c	c	c

2. General Crystallographic Characteristics.—The writer is indebted to both A. de Bretteville, Jr., and F. V. Ryer for experimental work resulting in the production of acceptable single crystals of sodium stearate and sodium palmitate.

The habit of the stearate crystals is well pictured and described by Thiessen and Stauff, although it should be mentioned that their labeling of axes in their figure 1 does not correspond with their labeling of measured cell edges. The crystals are extremely thin tabular parallel to {001} and at the same time elongated parallel to the b axis. The tablet is outlined by the pinacoid {100} and the prism {110}.

The habit of the palmitate crystals as grown from the alcohol solutions containing grains of sodium chloride is also very thin tabular parallel to {001}, but the outline of the tablet is diamond-shaped due to the exclusive development of {110}; the form {100} present in the stearate is missing in the palmitate. This variation in habit presumably is due to growth of the palmitate in the presence of an impurity, sodium chloride.

The following further observations were made specifically on the stearate, but apply, at least in part, to the palmitate also: The crystals have eminent {001} cleavage. They also have good {010} cleavage, although this is ordinarily overshadowed by the {001} cleavage. The plates are extremely plastic due to easy translation gliding with T = {001}, t = any direction in this plane, but [010] preferred; crystals bend easily, with a special tendency to bend with [100] as folding axis.

There is a structural relation between the possibility of bending crystals about a folding axis and the growth of crystals bent about the folding axis. A common kind of stearate crystal is an arborescent growth in which the main trunk crystal splits up, each split branching away from it by bending about the folding axis. These branches commonly bend away from the trunk, then bend back parallel with it, all the bending taking place about the folding axis [100]. In a crop of crystals grown from an alcoholic solution which has been allowed to stand some time, it is not uncommon to find a complexly branched individual occupying several square centimeters in total exposed bent and branched area, yet only as thick as the width of the trunk, a small fraction of a millimeter.

3. Symmetry.—In view of the difference between the results reported here and those previously reported by Thiessen and Stauff, they were checked by several methods.

For sodium stearate, equi-inclination Weissenberg photographs were taken of zero, first and second levels for rotation about the axis of elongation (Fig. 4). A set of equal-cone de Jong-Bouman photographs (Figs. 1 and 2) of the zero, first, second, third and fourth levels for the same rotation axis were also made.

All these photographs clearly indicate a level symmetry of C_2 (see Figs. 1, 2 and 4). This constitutes sufficient proof that the centrosymmetrical symmetry of the crystal is C_{2h}, $2/m$; consequently the crystal is monoclinic. This symmetry was established with sufficient data for the palmitate also. The axis of elongation of the stearate is consequently the b axis.

4. Space Lattice Type.—The photographs just discussed show that the $(010)^*$ planes are based upon a parallelogram net, but the reflections recorded by both Weissenberg and de Jong-Bouman photographs are so close together in the direction of the vector c^* that it is extremely difficult to be sure exactly how adjacent levels of the reciprocal lattice superpose. In order to resolve this difficulty and determine the correct stacking sequence, recourse was had to the moving film precession method[3] with [100] as the generator of the precession cone. This showed that the $(100)^*$ planes are based upon a diamond net and that these nets in adjacent levels are exactly superposed by the a^* translation. This information, taken with that given above, uniquely determines the lattice as A-centered monoclinic.

5. Space Group.—The only doubled translations in the reciprocal lattice are all a^* in central $(010)^*$. This indicates a glide a. The diffraction symbol of the crystal is therefore $2/mA-/a$. This is consistent with either of the two space groups Aa or $A2/a$.

6. Cell Dimensions.—The following cell dimensions were determined from the b-axis rotation photograph and various de Jong-Bouman and precession photographs (accuracy estimated at about $^1/_2\%$):

	Na stearate·$^1/_2$ H_2O	Na palmitate·$^1/_2$ H_2O
a	9.16 A	9.13 A
b	8.00	8.01
c	103.96	91.85
β	93° 43′	...

The matrix of the transformation from the axes given by Thiessen and Stauff to those of Buerger is

$$\begin{Vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{Vmatrix}$$

These crystals contain 16 hemihydrated molecules per unit cell. It is interesting to note that, if the crystals were anhydrous, as formerly sup-

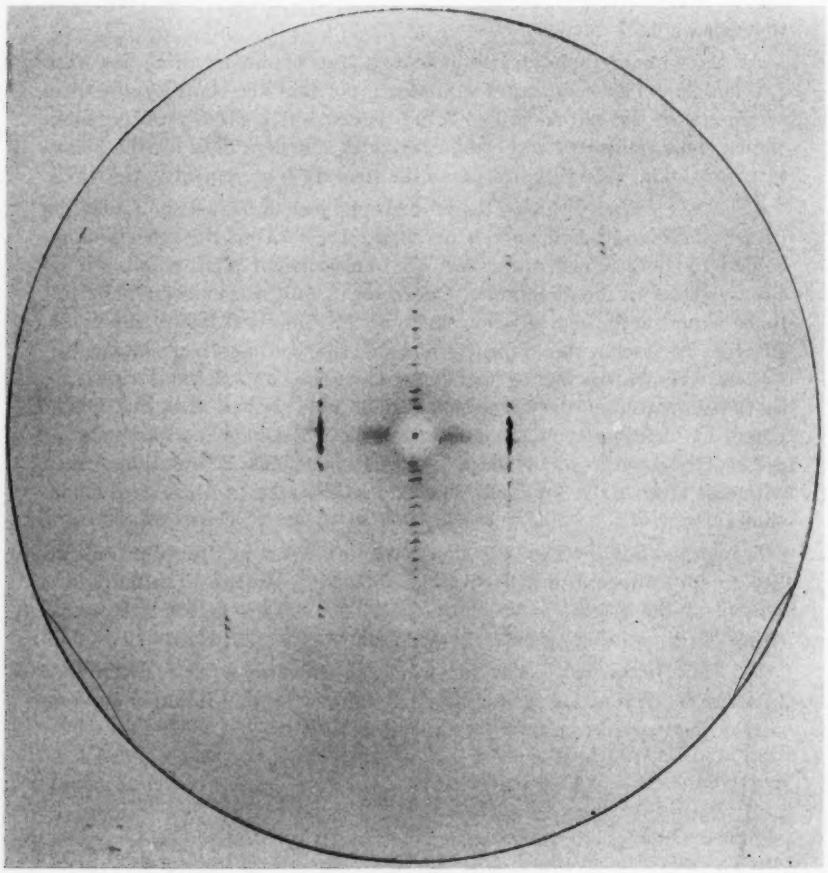

FIGURE 1

de Jong-Bouman photograph, $NaSt \cdot {}^1/_2H_2O$; zero level $(010)^*$ net. (Cu$K\alpha$ radiation.)

posed, this would require 16 chemically identical molecules in a unit cell whose space group provides, at most, an 8-fold position, a somewhat difficult situation to understand. The hemihydrate character, however, is equivalent to having to accommodate only eight groups of the type $2NaSt.H_2O$ in the cell, and the eight water molecules exactly fill the eight

general equipoints of the one of the two space group possibilities, $A2/a$. The alternative space group, Aa, is unlikely because it can provide only four equipoints for the eight waters.

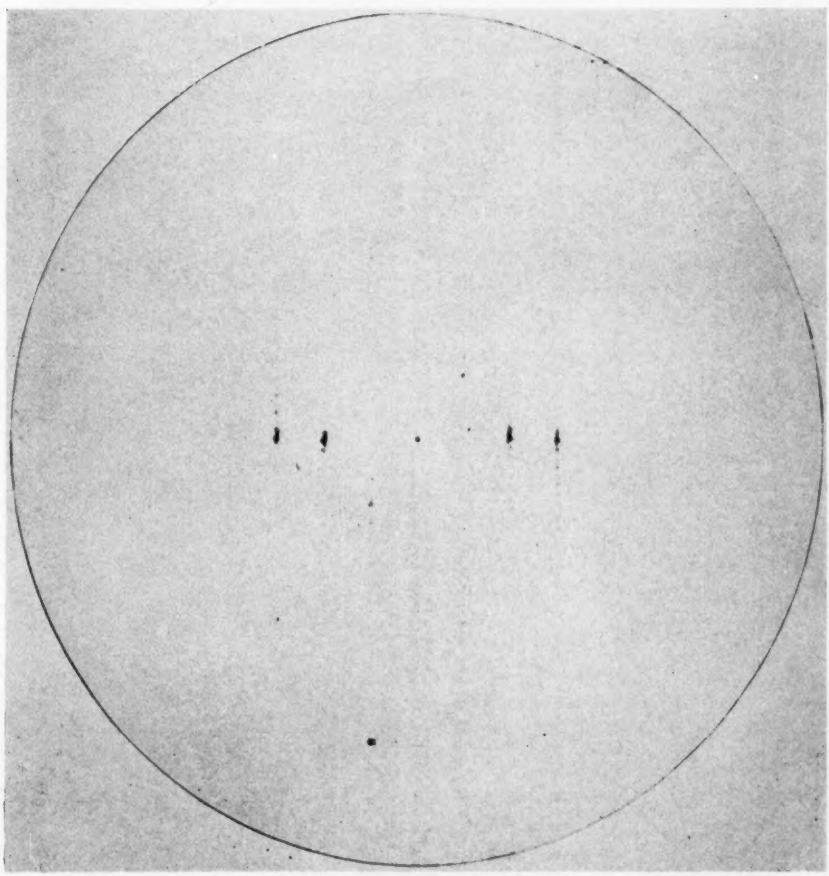

FIGURE 2

de Jong-Bouman photograph, $NaSt \cdot {}^1/_2H_2O$; second level (010)* net. (Cu$K\alpha$ radiation.)

7. *Indexing Powder Photographs of Soap Crystals.*—There are those who would attempt to assign indices to the lines of powder photographs of soap crystals. Even a casual comparison of Weissenberg, rotating crystal and powder photographs of any soap crystal makes it most apparent that each line on the powder photograph is actually a band composed of many re-

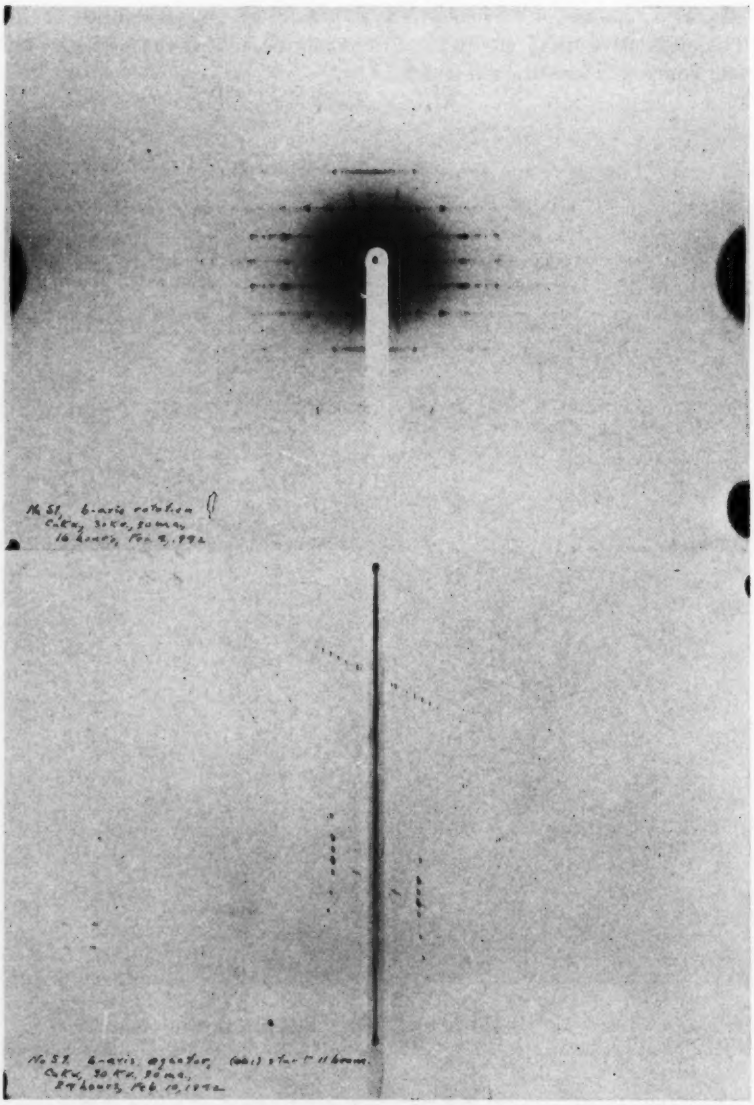

Figure 3 (above). Rotating-crystal photograph, $NaSt \cdot {}^1/_2H_2O$; b axis rotation.

Figure 4 (below). Weissenberg photograph, $NaSt \cdot {}^1/_2H_2O$; b axis rotation, zero level. Note the sets of spots nearly equidistant from the center line, especially near the bottoms of festoons. In the rotating-crystal photograph above, the spots of a set are unresolved and consequently the rotating-crystal photograph cannot be indexed.

flections of nearly identical sin θ value. Any indexing of powder photographs of soap crystals is consequently very questionable. It hardly need be said that no notion of symmetry can be obtained from powder photographs of crystals with such large cells as soaps.

There is also a prevailing opinion in some quarters that if the computed length of a soap molecule fits the spacing $d_{(001)}$, the crystal is orthorhombic, while if it must be tipped to fit the spacing, the crystal is monoclinic. Neither of these propositions is necessarily true.

[1] Buerger, M. J., Smith, L. B., Bretteville, A. de, Jr., and Ryer, F. V., "The Lower Hydrates of Soap," *Proc. Nat. Acad. Sci.*, **28**, 526–529 (1942).

[2] Thiessen, Peter A, and Stauff, Joachim, "Feinbau und Umwandlungen kristallisierter Alkalisalze langkettiger Fettsäuren," *Zeit. Physikal. Chem.* (*A*), **176**, 397–429 (1936).

[3] Buerger, M. J., "X-ray Crystallography," 210–211, John Wiley & Sons, Inc., New York, 1942.

STATISTICAL METRICS

BY KARL MENGER

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF NOTRE DAME

Communicated October 27, 1942

We shall call *statistical metric* a set S such that with each two elements ("points") p and q of S a probability function $\Pi(x;\ p, q)$ is associated satisfying the following conditions:

1. $\Pi(0;\ p, p) = 1$.
2. If $p \neq q$, then $\Pi(0;\ p, q) < 1$.
3. $\Pi(x;\ p, q) = \Pi(x;\ q, p)$.
4. $T[\Pi(x;\ p, q), \Pi(y;\ q, r)] \leq \Pi(x + y;\ p, r)$.

where $T(\alpha, \beta)$ is a function defined for $0 \leq \alpha \leq 1$ and $0 \leq \beta \leq 1$ such that

(*a*) $0 \leq T(\alpha, \beta) \leq 1$.
(*b*) T is non-decreasing in either variable.
(*c*) $T(\alpha, \beta) = T(\beta, \alpha)$.
(*d*) $T(1, 1) = 1$.
(*e*) If $\alpha > 0$, then $T(\alpha, 1) > 0$.

By a probability function we mean a non-decreasing function defined for all non-negative values of x, continuous to the right, with values between 0 and 1, and converging toward 1 as x increases beyond all bounds.

We call $\Pi(x;\ p, q)$ the *distance function* of p and q and interpret it as the probability that the points p and q have a distance $\leq x$. Condition 4,

our "triangular inequality," implies that $\Pi(z;\ p, r) \geq$ Max. $T[\Pi(x;\ p, q), \Pi(z - x;\ q, r)]$ for all points q and all numbers x between 0 and z. We shall call the function T the *triangular norm* of the statistical metric, and more specifically refer to the metric defined above as a T-metric. A triangular norm T will be called *simple* if

(*f*) $0 < T(\alpha, \beta) < 1$ for $0 < \alpha \cdot \beta < 1$.

An ordinary metric space is a statistical metric such that for each pair of points p, q there exists a number $d(p, q) \geq 0$ with the property that $\Pi(x;\ p, q)$ is $= 0$ if $x < d(p, q)$, and $= 1$ if $x \geq d(p, q)$.

On the basis of our postulates large parts of metric geometry can be developed, in particular, a theory of betweenness. We shall say that q lies *between* p and r (and we write pqr) if

$$T[1 - \Pi(x;\ p, q), 1 - \Pi(y;\ q, r)] \leq 1 - \Pi(x + y;\ p, r).$$

Equivalent is the assumption

$$\Pi(z;\ p, r) \leq 1 - \text{Max. } T[1 - \Pi(x;\ p, q), 1 - \Pi(z - x;\ q, r)] \text{for } 0 \leq x \leq z.$$

Obviously, if pqr then rqp. In metric spaces if q and r are distinct, then pqr and prq are incompatible. In a statistical metric we can only prove: If q and r are apart, then pqr and prq are incompatible, where q and r are said to be apart if there exists a number $y > 0$ such that $\Pi(y;\ q, r) = 0$.

If for each two points of a statistical metric S the distance function $\Pi(x;\ p, q)$ belongs to a family $\mathfrak{P}$ of probability functions, we call S metrized by means of $\mathfrak{P}$. Let $\mathfrak{P}$ be a 2 parameter family of probability functions $\Pi(x;\ a', a'')$ defined for all real numbers a' and a'' such that $0 \leq a' \leq a''$ and satisfying the conditions

$$\Pi(x;\ a', a'') = 0 \text{ if } x \leq a',$$
$$\Pi(x;\ a', a'') = 1 \text{ if } x \geq a'',$$
$$0 < \Pi(x;\ a', a'') < 1 \text{ for } a' < x < a''.$$

If S is metrized by means of $\mathfrak{P}$ and T is simple, then for each three points p, q, r one of which lies between the two other ones, two of the distance functions $\Pi(x;\ p, q)$, $\Pi(y;\ q, r)$, $\Pi(z;\ p, r)$ determine the third. In particular, if pqr and $\Pi(x;\ p, q) = \Pi(x;\ a', a'')$ and $\Pi(y;\ q, r) = \Pi(y;\ b', b'')$, then $\Pi(z;\ p, r) = \Pi(z;\ a' + b', a'' + b'')$. From this theorem one readily derives the classical law: If pqr and prs, then pqs and qrs.

If

$$\Pi(x;\ p, q) = \Pi(x;\ r, s) = \Pi(x;\ a', a'')$$
$$\Pi(x;\ q, r) = \Pi(x;\ p, s) = \Pi(x;\ b', b'')$$
$$\Pi(x;\ p, r) = \Pi(x;\ q, s) = \Pi(x;\ a' + b';\ a'' + b''),$$

then p, q, r, s form what may be called a *pseudo-linear statistical quadruple*, i.e., a quadruple which cannot be ordered by means of the between-relation though for each three of the four points one lies between the other two.

If a statistical T-metric S metrized by means of $\mathfrak{P}$ contains more than four points, then by virtue of the properties of betweenness this relation can be used to order S. Moreover, the other ideas of metric geometry (convexity, geodesics, etc.) can be applied.

The three principal applications of statistical metrics are to macroscopic, microscopic and physiological spatial measurements. Statistical metrics are designed to provide us (1) with a method removing conceptual difficulties from microscopic physics and transferring them into the underlying geometry, (2) with a treatment of thresholds of spatial sensation eliminating the intrinsic paradoxes of the classical theory. For a given point p_0 the number $\Pi(0; p_0, q)$ considered as a function of the point q indicates the probability that q cannot be distinguished from p_0. The study of this function should replace the attempt to determine a definite set of points q which cannot be distinguished from p_0. This function could also be used advantageously instead of a relation of physical identity for which, as Poincaré emphasized on several occasions, we always have triples p, q, r for which

$$p = q,\ q = r, \text{ and } p \neq r.$$

Experiments indicate that q sometimes can and sometimes cannot be distinguished from p_0. Hence, the adequate description of the situation seems to arise from counting the relative frequency of these occurrences.

NATURAL ISOMORPHISMS IN GROUP THEORY

BY SAMUEL EILENBERG AND SAUNDERS MACLANE

DEPARTMENTS OF MATHEMATICS, UNIVERSITY OF MICHIGAN AND HARVARD UNIVERSITY

Communicated October 26, 1942

1. Introduction.—Frequently in modern mathematics there occur phenomena of "naturality": a "natural" isomorphism between two groups or between two complexes, a "natural" homeomorphism of two spaces and the like. We here propose a precise definition of the "naturality" of such correspondences, as a basis for an appropriate general theory. In this preliminary report we restrict ourselves to the natural isomorphisms of group theory; with this limitation we can present the basic concepts of our theory without developing the axiomatic approach necessary for a general treatment applicable to various branches of mathematics.

Properties of character groups (see the definitions in § 5 below) may serve to illustrate the ideas involved. Thus, it is often asserted that the character group of a finite group G is isomorphic to the group itself, but not in a "natural" way. Specifically, if G is cyclic of prime order p, there is for each generator of G an isomorphism of G to its character group, so that the proof furnishes $p - 1$ such isomorphisms, no one of which is in any way distinguished from its fellows. However, the proof that the character group of the character group of G is isomorphic to G itself is considered "natural," because it furnishes for each G a unique isomorphism, not dependent on any choice of generators.

To give these statements a clear mathematical meaning, we shall regard the character group $Ch(G)$ of G as a function of a variable group G, *together* with a prescription which assigns to any homomorphism γ of G into a second group G',

$$\gamma : G \to G',$$

the induced homomorphism (see (5) below)

$$Ch(\gamma) : Ch(G') \to Ch(G).$$

The functions $Ch(G)$ and $Ch(\gamma)$ jointly form what we shall call a "functor"; in this case, a "contravariant" one, because the mapping $Ch(\gamma)$ works in a direction opposite to that of γ. A natural isomorphism between two functions of groups will be an isomorphism which commutes properly with the induced mappings of the functors.

With our description of a natural isomorphism, practically all the general isomorphisms obtained in group theory and its applications (homology theory, Galois theory, etc.) can be shown to be "natural." This results in added clarity in such situations. Furthermore, there are definite proofs where the naturality of an isomorphism is needed, especially when a passage to the limit is involved. In fact, our condition $(E2)$ below appears in the definition of the isomorphism of two direct or two inverse systems of groups.[1]

2. *Functors.*—The definition of a functor will be given for the typical case of a functor T which depends on two groups as arguments, and is *covariant* in the first argument and *contravariant* in the second. Such a functor is determined by two functions. The *group* function determines for each pair of topological groups G and H (contained in a given legitimate set of groups) another group $T(G, H)$. The *mapping function* determines for each pair of homomorphisms[2] $\gamma : G_1 \to G_2$ and $\eta : H_1 \to H_2$ a homomorphism $T(\gamma, \eta)$, such that

$$T(\gamma, \eta) : T(G_1, H_2) \to T(G_2, H_1). \qquad (1)$$

We require that $T(\gamma, \eta)$ be the identity isomorphism whenever γ and η are identities, and that, whenever the products $\gamma_2\gamma_1$ and $\eta_2\eta_1$ are defined,

$$T(\gamma_2\gamma_1, \eta_2\eta_1) = T(\gamma_2, \eta_1)T(\gamma_1, \eta_2). \tag{2}$$

Some functors will be defined only for special types of groups (e.g., for abelian groups) or for special types of homomorphisms (e.g., for homomorphisms "onto").

If γ and η are both isomorphisms,[3] it follows from these conditions that $T(\gamma, \eta)$ is also an isomorphism. Consequently, if the groups G_1 and G_2 and the groups H_1 and H_2 are isomorphic, the functor T gives rise to isomorphic groups $T(G_1, H_1)$ and $T(G_2, H_2)$.

3. *Examples.*—The direct product $G \times H$ of two groups may be regarded as the group function of a functor. The corresponding mapping function specifies, for each pair of homomorphisms $\gamma: G_1 \to G_2$ and $\eta: H_1 \to H_2$, an induced homomorphism $\gamma \times \eta$, defined for every element (g_1, h_1) in $G_1 \times H_1$ as

$$[\gamma \times \eta](g_1, h_1) = (\gamma g_1, \eta h_1).$$

Then

$$\gamma \times \eta : G_1 \times H_1 \to G_2 \times H_2, \tag{3}$$

and, whenever $\gamma_2\gamma_1$ and $\eta_2\eta_1$ are defined, one has

$$(\gamma_2\gamma_1) \times (\eta_2\eta_1) = (\gamma_2 \times \eta_2)(\gamma_1 \times \eta_1). \tag{4}$$

Except for the absence of contravariance, these conditions are parallel to (1) and (2), hence $G \times H$, $\gamma \times \eta$ define a functor, covariant in both G and H.

Whitney's tensor product[4] $G \circ H$ of two discrete groups[5] G and H is the group function of a functor. The elements of this group are all finite sums $\Sigma g_i \circ h_i$ of formal products $g_i \circ h_i$; the group operation is the obvious addition, and the relations are $g \circ (h + h') = g \circ h + g \circ h'$ $(g + g') \circ h = g \circ h + g' \circ h$. Given two homomorphisms $\gamma: G_1 \to G_2$ and $\eta: H_1 \to H_2$, there is an induced homomorphism $\gamma \circ \eta$ of $G_1 \circ H_1$ into $G_2 \circ H_2$, defined for any generator $g_1 \circ h_1$ of $G_1 \circ H_1$ as

$$[\gamma \circ \eta](g_1 \circ h_1) = (\gamma g_1) \circ (\eta h_1) \in G_2 \circ H_2.$$

Formulae (3) and (4), with the cross replaced by the circle, again hold, so that $G \circ H$, $\gamma \circ \eta$ determine a functor of discrete groups, covariant in both arguments.

In a similar fashion, the free product of two groups leads to a functor.

An important functor is given by the group of all homomorphisms ϕ of a fixed locally compact topological abelian group G into another topological abelian group H. The sum of two such homomorphisms ϕ_1 and ϕ_2 is de-

fined for each $g \in G$ by setting $(\phi_1 + \phi_2)(g) = \phi_1(g) + \phi_2(g)$. Under this operation, all $\phi: G \to H$ constitute a group $Hom(G, H)$; it carries an appropriate topology, the description of which we omit. For given $\gamma: G_1 \to G_2$ and $\eta: H_1 \to H_2$ and for each $\phi \in Hom(G_2, H_1)$ we have

$$G_1 \xrightarrow{\gamma} G_2 \xrightarrow{\phi} H_1 \xrightarrow{\eta} H_2.$$

Consequently we define $Hom(\gamma, \eta)(\phi) = \eta\phi\gamma$, and verify that

$$Hom(\gamma, \eta): Hom(G_2, H_1) \to Hom(G_1, H_2),$$

$$Hom(\gamma_2\gamma_1, \eta_2\eta_1) = Hom(\gamma_1, \eta_2)\, Hom(\gamma_2, \eta_1).$$

Clearly when γ and η are identity mappings of G and H the induced mapping $Hom(\gamma, \eta)$ is the identity mapping of $Hom(G, H)$ on itself. Hence the functions $Hom(G, H)$ and $Hom(\gamma, \eta)$ determine for abelian groups a functor Hom, covariant in H and contravariant in G.

The special case when H is the group P of reals modulo 1 furnishes the character group,

$$Ch(G) = Hom(G, P), \quad Ch(\gamma) = Hom(\gamma, e)$$

where e is the identity mapping of P on itself. Therefore the character group is a contravariant functor, defined for abelian groups. Explicitly, if we express the result $\chi(g)$ of applying the character χ to the element $g \in G$ as the value (a real number modulo 1) of the bilinear form (g, χ), the definition of $Ch(\gamma)$ can be written as

$$(g, Ch(\gamma)\chi') = (\gamma g, \chi'), \quad g \in G, \quad \chi' \in Ch(G'). \tag{5}$$

4. *Equivalence of Functors.*—Let T and S be two functors which are, say, both covariant in the variable G and contravariant in H. Suppose that for each pair of groups G and H we are given a homophorism

$$\tau(G, H): T(G, H) \to S(G, H).$$

We say that τ establishes a natural *equivalence* of the functor T to the functor S and that T is naturally *equivalent* to S (in symbols, $\tau: T \longleftrightarrow S$) whenever

(*E*1) Each $\tau(G, H)$ is a bicontinuous isomorphism of $T(G, H)$ onto $S(G, H)$;

(*E*2) For each $\gamma: G_1 \to G_2$ and $\eta: H_1 \to H_2$,
$\tau(G_2, H_1)T(\gamma, \eta) = S(\gamma,\eta)\tau\ (G_1, H_2)$.

The first requirement insures the term-by-term isomorphism of the two group functions $T(G, H)$ and $S(G, H)$, while the second requirement is

precisely the "naturality" condition. It can be shown that the condition ($E2$) is implied by two special cases; the case when η is an identity, and the case when γ is an identity.

This relation of natural equivalence between functors is reflexive, symmetric and transitive. In many cases we dispense with condition ($E1$), and obtain a more general concept of a "transformation" of a functor T into a functor S.

5. *Examples of Natural Equivalence.*—The well-known isomorphism

$$G \cong Ch(Ch(G)) \tag{6}$$

for locally compact abelian groups, can be regarded as an equivalence of functors, and is in this sense *natural*. The right-hand side of (6) suggests the covariant functor, Ch^2, defined by iteration of the functor Ch, as

$$Ch^2(G) = Ch(Ch(G)), \quad Ch^2(\gamma) = Ch(Ch(\gamma)).$$

The left-hand side of (6) suggests the identity functor, I,

$$I(G) = G, \quad I(\gamma) = \gamma.$$

The bilinear form $(g, \chi) = \chi(g)$ determines to each character $\chi \in Ch(G)$ and each $g \in G$ a real number modulo 1; similarly the form $(\chi, h) = h(\chi)$ is defined for each $h \in Ch^2(G)$. The form (g, χ), regarded as a function of χ for fixed g, is a character h in $Ch^2(G)$ which we call $[\tau(G)]g$. Explicitly, this definition of τ reads

$$(\chi, \tau(G)g) = (g, \chi), \quad g \in G, \quad \chi \in Ch(G).$$

The validity of condition ($E1$) for $\tau(G)$ is the basic theorem of character theory. The condition ($E2$) asserts that in the diagram

$$\begin{array}{ccc} G & \xrightarrow{\tau(G)} & Ch^2(G) \\ \downarrow{\scriptstyle \gamma} & & \downarrow{\scriptstyle Ch^2(\gamma)} \\ G' & \xrightarrow{\tau(G')} & Ch^2(G') \end{array}$$

the two paths leading from G to $Ch^2(G')$ have the same effect, or that, for each $g \in G$, both elements $\tau(G')\gamma g$ and $Ch^2(\gamma)\tau(G)g$ are identical as elements of $Ch^2(G')$. This means that, for each $\chi \in Ch(G')$, one should have

$$(\chi', \tau(G')\gamma g) = (\chi', Ch^2(\gamma)\tau(G)g).$$

By the definition of τ, the expression on the left is simply $(\gamma g, \chi')$. By successive application to the expression on the right of the definitions of Ch, τ and Ch, we obtain

$$(\chi', Ch^2(\gamma)\tau(G)g) = (Ch(\gamma)\chi', \tau(G)g) = (g, Ch(\gamma)\chi') = (\gamma g, \chi').$$

The identity of these results shows that we do have a natural equivalence $\tau(G): G \longleftrightarrow Ch^2(G)$.

When G is finite, the isomorphism $G \to Ch\, G$ cannot be "natural" according to our definitions, for the simple reason that the functor I on the left is covariant, while the functor Ch on the right is contravariant.

As other examples of equivalences between functors, we may cite the usual isomorphisms which give the associative and commutative laws for the direct product, the tensor product and the free product. Various distributive laws, such as

$$(G_1 \times G_2) \circ H \cong (G_1 \circ H) \times (G_2 \circ H),$$

$$Hom(G_1 \times G_2, H) \cong Hom(G_1, H) \times Hom(G_2, H),$$

when established with the obvious isomorphisms, are in fact equivalences between functors.

A less obvious relation between the tensor product and the functor "*Hom*" is[6]

$$Hom(G, Hom(H, K)) \cong Hom(G \circ H, K), \tag{7}$$

where G and H are discrete abelian groups, K a topological abelian group. This isomorphism is obtained by a correspondence $\tau(G, H, K)$ which specifies for each element $\phi \in Hom(G, Hom(H, K))$ a corresponding homomorphism in $Hom(G \circ H, K)$, defined for any generator $g \circ h$ of $G \circ H$ as

$$[\tau(G, H, K)]\,(\phi)(g \circ h) = [\phi(g)](h) \text{ in } K.$$

One may show that τ does give an isomorphism, bicontinuous in the appropriate topologies. Both sides of (7) may be treated as the group functions of functors which are obtained by composition from "*Hom*" and "$\circ$." The corresponding mapping functions, for given homomorphisms

$$\gamma: G_1 \to G_2, \quad \eta: H_1 \to H_2, \quad \kappa: K_1 \to K_2,$$

are defined by a parallel composition as

$$Hom(\gamma, Hom(\eta, \kappa)), \quad Hom(\gamma \circ \eta, \kappa).$$

Both functors are contravariant in G and H, covariant in K.

The naturality condition for the isomorphism τ reads

$$\tau(G_1, H_1, K_2)\, Hom(\gamma, Hom(\eta, \kappa)) = Hom(\gamma \circ \eta, \kappa)\tau(G_2, H_2, K_1).$$

Both sides, when applied to an element $\phi \in Hom(G_2, Hom(H_2, K_1)$ yield a homomorphism in $Hom(G_1 \circ H_1, K_2)$. If each of these homomorphisms is applied to a typical generator $g_1 \circ h_1$ of the tensor product $G_1 \circ H_1$, straightforward application of the relevant definitions shows that the same element of K_2 is obtained in both cases; namely, $\kappa\{[\phi(\gamma(g_1))](\eta(h_1))\}$.

One may also see directly that this expression represents the only way of constructing an element of K_2 from the elements g_1 and h_1 and the mappings κ, ϕ, γ and η.

The natural isomorphism (7) has some interesting consequences. If K is taken to be the group P of real numbers modulo 1, $Hom(H, K)$ becomes the character group $Ch(H)$, and the formula may be written as

$$Hom(G, Ch\, H) \cong Ch(G \circ H).$$

Applying the functor Ch to both sides and using the natural equivalence of Ch^2 and I, we obtain the equivalence

$$G \circ H \cong Ch\, Hom(G, Ch\, H).$$

Since this is "natural," this could be used as a definition of the tensor product $G \circ H$.

6. *Generalizations.*—With the appropriate definition of a normal subfunctor S of a functor T one can construct a quotient functor T/S, whose group function has as its values quotient groups (i.e., factor groups). With this operation, all the standard constructions on groups may be represented as group functions of suitable functors.

An inspection of the concept of a functor and of a natural equivalence shows that they may be applied not only to groups with their homomorphisms, but also to topological spaces with their continuous mappings, to simplicial complexes with their simplicial transformations, and to Banach spaces with their linear transformations. These and similar applications can all be embodied in a suitable axiomatic theory. The resulting much wider concept of naturality, as an equivalence between functors, will be studied in a subsequent paper.

[1] Pontrjagin, L., "Ueber den algebraischen Inhalt der topologische Dualitätsätze," *Mathematische Ann.*, **105**, 165–205 (1931). Lefschetz, S., "Algebraic Topology," *Am. Math. Soc. Colloquium Pub.*, **27**, 55 (1942).

[2] By a homomorphism we mean a *definite* pair of groups G_1 and G_2 and a (continuous) homomorphic mapping γ_1 of the first onto a subgroup of the second. The product $\gamma_2\gamma_1$ is defined for those pairs γ_1: $G_1 \longrightarrow G_2$, γ_2: $G_2' \longrightarrow G_3$ with $G_2 = G_2'$.

[3] By an isomorphism we mean a homomorphism of G_1 *onto* G_2 which is one-one and bicontinuous.

[4] Whitney, H., "Tensor Products of Abelian Groups," *Duke Math. Jour.*, **4**, 495–528 (1938).

[5] Here and subsequently the group operation in G and in H is written as addition, whether or not the groups are abelian.

[6] This isomorphism was established by the authors; cf. *Ann. Math.*, **44** (1943).

CONCERNING INTERSECTING CONTINUA

BY R. L. MOORE

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF TEXAS

Communicated November 2, 1942

In this paper a study is made of certain relationships involving (1) a continuum M, (2) its boundary β, (3) the components of $M - \beta$, (4) the components of the complements of the closures of the components of $M - \beta$ and (5) the common part of M and another continuum K. Numbered axioms and chapters referred to are axioms and chapters of the author's book, "Foundations of Point Set Theory."[1]

THEOREM 1. *If, in a space satisfying Axioms 0 and 1, M is a compact continuum, with a boundary β such that, if D is a component of $M - \beta$, $\beta \cdot \overline{D}$ is connected, and K is a continuum intersecting β, then $M \cdot K$ is a continuum.*

Proof. By Theorem 3 of my paper, "Concerning a Continuum and Its Boundary,"[2] β is connected. Hence so is $K + \beta$. If $M \cdot K$ is K or a subset of β then $M \cdot K + \beta$ is $K + \beta$ or β. If $M \cdot K$ is neither K nor a subset of β then $(K + \beta) - \beta$ is the sum of two mutually separated point sets $M \cdot K - M \cdot K \cdot \beta$ and $K - M \cdot K$. Hence $(M \cdot K - M \cdot K \cdot \beta) + \beta$ is connected. But this point set is identical with $M \cdot K + \beta$.

Theorem 1 does not remain true if, in the statement of its hypothesis, the stipulation that M is compact is replaced by the stipulation that K is compact even though there is added, to this hypothesis, the additional stipulation that the entire boundary of every component of $M - \beta$ is a connected subset of β and the conclusion is so weakened as to require only that there exists a continuum containing $M \cdot K$ and lying in $M \cdot K + \beta$. This may be seen with the help of Example 2 of C. B. However if, in the above proof of Theorem 1, Theorem 3 of C. B. is replaced by Theorem 4 of that paper, the resulting argument establishes the following result.

THEOREM 2. *Theorem 1 remains true if, in the statement of its hypothesis, the requirement that M be compact is replaced by the requirement that Axiom 2 hold true.*

THEOREM 3. *If, in a space satisfying Axioms 0 and 1, the continuum K contains β, the boundary of the compact continuum M and, for every component D of $M - \beta$, $K \cdot \overline{D}$ is a continuum, then $M \cdot K$ is a continuum.*

THEOREM 4. *Theorem 3 remains true if the requirement that M be compact is replaced by the requirement that space satisfy Axiom 2.*

Theorems 3 and 4 may be easily established with the aid of Theorems 1 and 2, respectively, of C. B. Theorem 3 may also be easily proved with the help of Theorem 9 of my paper, "Concerning Accessibility."[3]

It may be seen from Example 2 of C. B. that Theorem 3 does not remain true if K instead of M is required to be compact.

THEOREM 5. *If, in a space satisfying Axioms 0 and 1, M is a continuum, K is a compact continuum intersecting M and, for every component E of $S - M$ that intersects K, T_E is a connected point set intersecting $M \cdot K$ such that, for every component Q of $K \cdot E$, whose closure intersects M, T_E intersects every component of $\overline{Q} \cdot M$ and T is the sum of the point sets T_E for all E's then $M \cdot K + T$ is connected.*

Proof. Suppose $M \cdot K + T$ is the sum of two mutually separated point sets F_1 and F_2. Every component of T intersects $M \cdot K$ and lies either in F_1 or in F_2. Hence $F_1 \cdot M \cdot K$ and $F_2 \cdot M \cdot K$ exist. Since these point sets are mutually separated and their sum is the closed set $M \cdot K$, therefore they are closed. Hence $K - M \cdot K$ contains a connected point set L such that $\overline{L}$ intersects F_1 and F_2. Let W denote the component of $S - M$ which contains L. The point set T_W is a connected subset of T intersecting both F_1 and F_2. This involves a contradiction.

Theorem 5 does not remain true if the stipulation that K is compact is replaced by the stipulation that M is compact. Consider the example obtained by interchanging M and K in Example 2 of C. B. But the following theorem holds.

THEOREM 6. *Theorem 5 remains true if the requirement that K be compact is replaced by the requirement that space satisfy Axiom 2.*

Proof. Suppose $M \cdot K + T$ is the sum of the mutually separated point sets F_1 and F_2. By a modification of Theorem 6 of my paper, "Concerning Domains Whose Boundaries Are Compact,"[4] there exists a component W of $S - M$, intersecting K, whose boundary intersects each of the closed point sets $F_1 \cdot M \cdot K$ and $F_2 \cdot M \cdot K$. As in the above proof of Theorem 5, this leads to a contradiction.

The following theorem easily follows with the help of Theorem 6.

THEOREM 7. *If, in a space satisfying Axioms 0, 1 and 2, the continuum K contains the boundary of the continuum M and every component of $S - M$ has a connected boundary then $M \cdot K$ is connected.*

The following theorem may be easily established with the help of Theorem 5.

THEOREM 8. *If, in a space satisfying Axioms 0 and 1, M is a continuum and K is a compact continuum intersecting M and every component of $S - M$ that intersects K has a connected boundary then $M \cdot K$ plus the boundaries of all components of $S - M$ that intersect K is connected.*

Theorem 8 does not remain true if its conclusion is replaced by "$M \cdot K$ is a subset of a component of $M \cdot K$ plus the boundary of M." Indeed it does not remain true if its hypothesis is strengthened by the addition of the stipulation that the boundary of every component of $S - M$ is connected, and its conclusion is replaced by "$M \cdot K$ is a subset of some compact subcontinuum of M." Consider the following example.

EXAMPLE 1. In a Cartesian plane E let T denote a totally disconnected compact and perfect point set lying on OY and such that the ordinate of its highest point is -10. Let A and B denote the points $(-10, 0)$ and $(10, 0)$, respectively. Let H denote the set of all points of the graph of $y = \sin(1/x)$ whose abscissae are neither less than $-1/\pi$ nor greater than $1/\pi$. Let N denote the set of all points of OX that lie between A and $(-1/\pi, 0)$ or between B and $(1/\pi, 0)$. Let M denote the point set $O + H + N + A + B$ and let K denote the sum of all straight line intervals with one end-point at one of the points A and B and the other one at a point of T. Let S denote $M + K$. Let Σ denote the subspace of E whose points are the points of S. Here β consists of two points, A and B.

However, the following theorem holds true.

THEOREM 9. *If, in a space satisfying Axioms 0 and 1, β is the boundary of the continuum M and K is a compact continuum intersecting M, and the common part of M and the boundary of each component of $S - M$ that intersects K is connected, then $M \cdot K$ is a subset of a component of $M \cdot K + \beta$.*

THEOREM 10. *If, in a space satisfying Axioms 0 and 1, K is a compact continuum intersecting the continuum M and, for every component E of $S - M$ that intersects K, $K \cdot \bar{E} \cdot M$ is a subset of some component of $M \cdot K$ then $M \cdot K$ is connected.*

Proof. For each component E of $S - M$ that intersects K, let T_E denote the component of $M \cdot K$ that contains $K \cdot \bar{E} \cdot M$ and let T denote the sum of all T_E's. By Theorem 5, $M \cdot K + T$ is connected. But it is identical with $M \cdot K$.

It may be seen from the following example that Theorem 10 does not remain true if the requirement that K be compact is omitted even though the resulting hypothesis is strengthened by the addition of the stipulation that (1) K contains β and (2) every component of $S - M$ is bounded by a connected subset of M and so is every component of $M - \beta$ and, furthermore, if D is a component of $M - \beta$, every component of $S - \bar{D}$ is so bounded.

EXAMPLE 2. In a Cartesian plane E, let A and D denote the points $(0, 2)$ and $(0, 1)$ and, for each n, let A_n and B_n denote the points $(1/n, 0)$ and $(-1/n, 0)$, respectively, and let AA_n and AB_n denote straight line intervals with end-points as indicated. Let M and K denote $D + AB_1 + AB_2 + \ldots$ and $D + AA_1 + AA_2 + \ldots$, respectively. Let S denote $M + K$. Let Σ denote the subspace of E whose points are the points of S.

Theorem 10 does not remain true if M instead of K is required to be compact. To see this, interchange M and K in the description of Example 2 of C. B.

Of Theorems 5–10, Theorem 7 is the only one that remains true if, in its statement, $S - M$ is replaced by $M - \beta$.

The following theorem may be easily proved.

THEOREM 11. *If, in a space satisfying Axiom 0, the closed point set M intersects the connected point set K and $M \cdot K$ is the sum of two mutually separated point sets H and L then both H and L intersect the boundary of M.*

THEOREM 12. *If, in a space satisfying Axioms 0 and 1, the compact point set β is the boundary of the compactly connected continuum M and, for every component D of $M - \beta$ and component E of $S - \overline{D}$, the boundary of E is a subset of a component of β then if K is a compact continuum intersecting M, $M \cdot K$ is a subset of a component of $M \cdot K + \beta$.*

Proof. Let N denote the point set obtained by adding together all components of $M \cdot K + \beta$ that intersect $M \cdot K$. Since $M \cdot K$ and β are closed and compact, N is closed. Suppose it is the sum of two mutually exclusive closed point sets H and L. These point sets both intersect $M \cdot K$. Since the continuum K is compact it contains a connected point set T lying in $S - M$ and such that $\overline{T}$ intersects both $H \cdot M \cdot K$ and $L \cdot M \cdot K$. Let A and B denote points belonging, respectively, to the subsets $\overline{T} \cdot H \cdot M \cdot K$ and $\overline{T} \cdot L \cdot M \cdot K$ of β. There exists a compact continuum M' lying in M and containing A and B. Suppose P is a point of $M' - M' \cdot \beta$. Let D_P denote the component of $M - \beta$ that contains P and let E_P denote the component of $S - \overline{D}_P$ that contains T. By hypothesis, the boundary of E_P is a subset of a component β_P of β. For each point P of $M' - M' \cdot \beta$, β_P shields $A + B$ from P in M. Hence, by Theorem 6 of C. B., there is a continuum containing A and B and lying in β. This involves a contradiction.

THEOREM 13. *Theorem 12 remains true if the requirement that K be compact is replaced by the requirement that $M \cdot K$ be compact and that space satisfy Axiom 2.*

Theorem 13 may be proved by an argument identical with that given to prove Theorem 12 except for the substitution of "By Theorem 6 of D. C. B., there exists" for "Since the continuum K is compact it contains" in the third sentence of that argument.

Theorem 12 does not remain true on the omission of the stipulation that M is compactly connected even though it be stipulated that K contains β. Consider the following example.

EXAMPLE 3. In a Cartesian plane E, let O denote the origin and, for each positive integer n, let A_n, B_n, C_n, D_n and E_n denote the points $(0, 1/n)$, $(1/n, 1/n)$, $(1/n, -1/n)$, $(-1/n, -1/n)$ and $(-1/n, 1)$ and let F_n denote a point lying midway between B_n and A_{n+1}. If X and Y are two points let XY denote the straight line interval whose extremities are X and Y. Let t_n denote the point set $A_nB_n + B_nC_n + C_nD_n + D_nE_n$. Let M and K denote the point sets $O + t_1 + t_2 + t_3 + \ldots$ and $O + A_1F_1 + F_1A_2 + A_2F_2 + F_2A_3 + \ldots$, respectively. Let S denote the point set $M + K$ and let Σ denote the subspace of E whose points are the points of S. Here β is the point set $O + A_1 + A_2 + \ldots$ and K is a compact con-

tinuum containing β. Furthermore if D is a component of $M - \beta$ then, for some n, D is $t_n - A_n$ and if E is a component of $S - \overline{D}_n$ then E is $S - t_n$ and its boundary is the point A_2 which is a connected subset of $\beta \cdot \overline{D}$. But $M \cdot K = \beta$ and every component of $M \cdot K + \beta$ is a point of the infinite point set β. Hence $M \cdot K$ is not a subset of any component of $M \cdot K + \beta$.

Neither does Theorem 12 hold true if the stipulation that β is compact is replaced by the stipulation that Axiom 2 holds true. Consider the following example.

EXAMPLE 4. In a Cartesian space E of three dimensions, for each positive integer n, let T_n denote the portion of the graph of $y = -n + 1/(x - x^2)$ that lies in the XY plane between the planes $x = 0$ and $x = 1$. For each n, let S_n denote the set of all points (x, y, z) of this graph such that $0 < x < 1$ and $0 < z < 1/n$. Let A, B and C denote the points (0, 0, 2), (1, 0, 2) and (1, 0, 0), respectively. Let K denote the arc obtained by adding together the straight line intervals OA, AB and BC. Let M denote the XY plane and let S denote $K + M + S_1 + S_2 + S_3 + \ldots$. Let Σ denote the subspace of E whose points are the points of S. Here K is compact and $M \cdot K = O + C$. The boundary of M is the point set obtained by adding together the open curves T_1, T_2, T_3, ..., the Y-axis and a line through C parallel to the Y-axis. The point set $M \cdot K + \beta$ is identical with β and no component of β contains both O and C. It is to be noted that β is not the sum of two mutually separated point sets containing O and C, respectively.

However, the following theorem holds true.

THEOREM 14. *If, in a space satisfying Axioms 0, 1, and 2, the continuum K contains β, the boundary of the continuum M, and, for every component D of $M - \beta$ and component E of $S - \overline{D}$, the boundary of E is connected then $M \cdot K$ is connected.*

Proof. Suppose $M \cdot K$ is the sum of two mutually exclusive closed point sets H and L. By Theorem 11, $H \cdot \beta$ and $L \cdot \beta$ exist. Furthermore, by hypothesis, β is a subset of their sum. Hence, by Theorem 5 of C. B., there exists a component D of $M - \beta$ such that $\overline{D}$ intersects both H and L. The common part of the continua $\overline{D}$ and K is the sum of the two mutually exclusive closed point sets $D \cdot K \cdot H$ and $\overline{D} \cdot K \cdot L$. Therefore, by Theorem 6 of D. B. C., there exists an arc AB from the point A of $\overline{D} \cdot K \cdot H$ to the point B of $\overline{D} \cdot K \cdot L$ and having no point in common with $\overline{D}$ except its end-points A and B. Let E denote the component of $S - \overline{D}$ which contains $AB - (A + B)$. The boundary of E is a connected subset of β containing the points A and B. This involves a contradiction.

THEOREM 15. *If, in a space satisfying Axioms 0 and 1, β is the boundary of the continuum M, D is a component of $M - \beta$, K is a compact continuum intersecting $\overline{D}$ but not lying wholly in it and Q is a component of $K \cdot (S - \overline{D})$ then every component of $\overline{Q} \cdot \overline{D}$ contains a point of β.*

Proof. Suppose the component H of $\overline{Q}\cdot\overline{D}$ contains no point of β. Let O denote some point of $Q - H$. Since $\overline{Q}$ is a compact continuum there exists a domain W containing H, but no point of $O + \beta$, and such that its boundary γ contains no point of $\beta + \overline{Q}\cdot\overline{D}$. The point set $\overline{Q}\cdot W$ contains a connected point set T such that $\overline{T}$ intersects γ and contains a point P of $\overline{D}$. Since P does not belong to β, it belongs to D. But $T + P$ is a connected point set containing no point of β. Hence $T + P$ is a subset of D. Since T is a subset of Q this involves a contradiction.

THEOREM 16. *If, in a space satisfying Axioms 0 and 1, β is the boundary of the compact continuum M and D is a component of $M - \beta$ and, for every component E of $S - \overline{D}$, the common part of β and the boundary of E is connected and K is a compact continuum intersecting $\overline{D}$ then $K\cdot\overline{D}$ is a subset of a component of $K\cdot\overline{D} + \beta$.*

Proof. For every component E of $S - \overline{D}$ that intersects K, let T_E denote the common part of β and the boundary of E. With the help of Theorem 15 it may be seen that if Q is a component of $K\cdot E$ every component of $Q\cdot D$ intersects β and therefore T_E. Hence, by Theorem 5, if T is the sum of the continua T_E for all E's, $K\cdot\overline{D} + T$ is connected. But T is a subset of β.

THEOREM 17. *If, in a space satisfying Axioms 0 and 1, β is the boundary of the compact continuum M and, for every component D of $M - \beta$ and component E of $S - \overline{D}$, the common part of β and the boundary of E is connected and K is a compact continuum containing β then $M\cdot K$ is a continuum.*

Proof. If D is a component of $M - \beta$, the continuum K intersects $\overline{D}$ and therefore, by hypothesis and Theorem 16, $K\cdot\overline{D}$ is a subset of a component T_D of $K\cdot\overline{D} + \beta$. Let T denote the sum of all the point sets T_D for all D's. By Theorem 1 of C. B., $\beta + T$ is connected. But this point set is identical with $M\cdot K$.

Theorem 17 remains true if the stipulation that the common part of β and the boundary of E is connected is replaced by the stipulation that it is a subset of a component of $M\cdot K$. But it does not remain true if "the common part of β and the boundary of E" is replaced by "the common part of $\overline{D}$ and the boundary of E" or by "the boundary of E." Consider the following example.

EXAMPLE 5. In a Euclidean plane E let α denote a definite square and let γ denote a definite square enclosed by α. Let O denote the midpoint of one side of γ and let Q denote a totally disconnected perfect point set lying on the opposite side of γ. Let M denote the point set obtained by adding together all straight line intervals with one end-point at O and the other one at a point of Q. Let K denote the sum of α and γ and the set of all points that lie between them. Let S denote $M + K$ and let Σ denote the subspace of E whose points are the points of S. Here S is compact, β is $O + Q$, K contains β and, for every component D of $M - \beta$,

$S - \overline{D}$ is connected and its boundary is $\overline{D}$. But $M \cdot K$ is the uncountable totally disconnected point set β.

Theorem 17 does not remain true if the requirement that K be compact is omitted. To see this, interchange M and K in the description of Example 2 of C. B. It is clear from Example 3 that it does not remain true on the omission of the requirement that M be compact.

[1] *Amer. Math. Soc. Colloquium Pub.*, **13**, New York (1932). The letter S denotes the set of all points.

[2] These PROCEEDINGS, **28**, 550–555 (1942). This paper will be referred to as "C. B."

[3] These PROCEEDINGS, **25**, 56–58 (1942).

[4] These PROCEEDINGS, **28**, 555–561 (1942). This paper will be referred to as "D. C. B." If the proof of Theorem 6 of this paper is modified only to the extent of replacing the definition of H_n by the requirement that (1) for each n greater than 1, H_n be defined as it is there defined for *each* n and (2) H_1 be a collection of connected domains properly covering K such that each of them lies in some region of G_1 and every one of them that intersects M intersects $M \cdot K$ and no one of them that intersects H has a point in common with any one of them that intersects L, then the arc C necessarily lies, except for its end-points, wholly in some component of S-M that contains points of K.

CONCERNING A CONTINUUM AND ITS BOUNDARY

BY R. L. MOORE

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF TEXAS

Communicated October 23, 1942

In this paper some theorems will be established concerning certain relationships between the boundary of a continuum and the components of that continuum minus its boundary. Numbered axioms and chapters herein referred to are axioms and chapters of the author's book "Foundations of Point Set Theory."[1]

THEOREM 1. *If, in a space satisfying Axioms 0 and 1, β is the boundary of the compact continuum M and, for every component D of $M - \beta$, T_D is a connected point set lying in M and containing the common part of β and the boundary of D, and T is the sum of all the point sets T_D for all such D's, then $\beta + T$ is connected.*

Proof. Suppose $\beta + T$ is the sum of two mutually separated point sets H and K. The point sets $H \cdot \beta$ and $K \cdot \beta$ exist and are mutually exclusive and closed. There exists a subcontinuum N of M which is irreducible from $H \cdot \beta$ to $K \cdot \beta$. The connected point set $N - (H \cdot \beta + K \cdot \beta)$ is a subset of some component L of $M - \beta$. Each of the point sets $H \cdot \beta$ and $K \cdot \beta$ con-

tains a limit point of $N - (H \cdot \beta + K \cdot \beta)$ and therefore of L. Hence T_L intersects both $H \cdot \beta$ and $K \cdot \beta$. This involves a contradiction.

THEOREM 2. *Theorem 1 remains true if, in the statement of its hypothesis, the requirement that M be compact is replaced by the requirement that space satisfy Axiom 2.*

Proof. Suppose $\beta + T$ is the sum of two mutually separated point sets H_1 and H_2. Suppose D is a component of $M - \beta$. Since the connected point set T_D is a subset of $H_1 + H_2$, it is a subset either of H_1 or of H_2. If there exists at least one component D of $M - \beta$ such that T_D is a subset of H_i, let H_i' denote H_i plus the sum of all such point sets D. Otherwise let H_i' denote H_i. Since Axiom 2 holds true the components of $M - \beta$ are all domains and, by Theorem 2(b) of Chapter II, every limit point of their sum belongs either to one of them or to the closure of the sum of their boundaries. But, since they are domains, no one of them contains a limit point of its complement, and the boundary of every one of them is a subset of β. It follows that if the point P of H_2' is a limit point of H_1' then P belongs to H_2 and is a limit point of H_1. This is impossible. Similarly H_1' contains no limit point of H_2'. Thus H_1' and H_2' are mutually separated. But their sum is the continuum M. This involves a contradiction.

THEOREM 3. *If, in a space satisfying Axioms 0 and 1, β is the boundary of the compact continuum M and, for every component D of $M - \beta$, the common part of β and the boundary of D is connected, then β is connected.*

THEOREM 4. *If, in a space satisfying Axioms 0, 1 and 2, β is the boundary of the continuum M and, for every component D of $M - \beta$, the common part of β and the boundary of D is connected, then β is connected.*

Theorems 3 and 4 are corollaries of Theorems 1 and 2, respectively.

If Axioms 0, 1 and 2 hold true and D is a component of the continuum M minus its boundary β then D is a domain and its boundary is a subset of β. Hence *Theorem 4 remains true if, in the statement of its hypothesis, "the common part of β and the boundary of D" is replaced by "the boundary of D."* But this is not true of Theorem 3 even though the resulting hypothesis is strengthened by the addition of the stipulation that space is compact. Consider the following example.

Example 1. In a Cartesian plane E let Q denote a compact totally disconnected closed point set lying on the Y-axis, let A and B denote the points $(-1, 0)$ and $(-2, 0)$, respectively, let S denote the sum of the straight line interval AB and all straight line intervals with one end-point at A and mid-point at a point of Q, let M denote the sum of all straight-line intervals with one end -point at A and the other one at a point of Q and let β denote the point set $A + Q$. Let Σ denote the subspace of E whose points are the points of S. In the compact space Σ, the boundary of the continuum M is β and if D is a component of $M - \beta$ then the boundary of D is the continuum $\bar{D}$. But β is not connected.

That Theorem 3 does not remain true if the stipulation that M is compact is replaced by the stipulation that β is compact may be seen from the following example.

Example 2. In a Cartesian plane E, let O and A denote the points (0, 0) and (0, 2), respectively, and, for each n, let B_n denote $(-1/n, 0)$. Let M denote the sum of O and the straight-line intervals AB_1, AB_2, AB_3,...let K denote a semi-circle with extremities at A and O and containing the point (1, 1). Let S denote $M + K$. Let Σ denote the subspace of E whose points are the points of S. Here β is $A + O$ and, for each component D of $M - \beta$, $\beta \cdot \bar{D}$ is A.

THEOREM 5. *If, in a space satisfying Axioms 0, 1 and 2, β, the boundary of the continuum M is the sum of two mutually exclusive closed point sets β_1 and β_2 then there is a component of $M - \beta$ whose closure intersects both β_1 and β_2.*

Proof. The closure of every component of $M - \beta$ intersects β and therefore either β_1 or β_2. Let M_i denote the set of all points that belong either to β_i or to some component of $M - \beta$ whose closure intersects β_i. Since their sum is the connected point set M, M_1 and M_2 are not mutually separated. Suppose they are mutually exclusive. Then one of them contains a limit point of the other one. But every component of $M - \beta$ is a domain and no domain contains a limit point of its complement. Hence either β_1 contains a limit point of M_2 or β_2 contains a limit point of M_1. Suppose β_1 contains a point P which is a limit point of M_2. Let Q denote the collection of all components of $M - \beta$ whose closures intersect β_2. Since it is not a limit point of β_2, P must be a limit point of Q^*. Hence, by Theorem 2(b) of Chapter II, P belongs to the closure of the sum of the boundaries of the domains of the collection Q. Since the boundary of every domain of Q belongs to the closed point set β_2 it follows that P belongs to β_2 and therefore that β_1 intersects β_2. The same result is obtained in case β_2 contains a limit point of β_1. Thus the supposition that M_1 and M_2 are mutually exclusive leads to a contradiction. It follows that there is a component of $M - \beta$ whose closure intersects β_1 and β_2.

Definition. If H, L and N are subsets of the continuum M, N is said to *shield H from L in M* if N has no point in common with L but intersects every connected subset of M which intersects both H and L.

THEOREM 6. *If A and B are points of the continuum M, T is a compact subcontinuum of M containing A and B, H is a closed and compact subset of M and, for each point P of $T - (A + B + T \cdot H)$ there is a continuum lying in H and shielding $A + B$ from P in M then A and B belong to the same component of H.*

Proof. Suppose A and B do not lie in the same component of H. There exists a well-ordered sequence α whose terms are the points of $T - (A + B + T \cdot H)$, a well-ordered sequence β_1 whose terms are the continua which

are subsets of H, and a well-ordered sequence β_2 whose terms are the continua that lie in $T + H$. Suppose T is not a subset of H. There exists a well-ordered sequence γ such that (1) T is the first term of γ, (2) each term of γ is a continuum lying in $T + H$ but not wholly in H and (3) if x is a term of γ and P_x is the first point of α belonging to x and H_x is the first term of β_1 which shields $A + B$ from P_x in M, T_{Ax} is the first term of β_2 which is an irreducible continuum from A to H_x lying in x and T_{Bx} is the first term of β_2 which is an irreducible continuum from B to H_x lying in x, then x is the last term of γ or $H_x + T_{Ax} + T_{Bx}$ is the first term following x in γ, according as $T_{Ax} + T_{Bx}$ is, or is not, a subset of H, (4) if γ' is a well-ordered proper subsequence of γ with no last term and γ' has, as one of its terms, each term of γ which, in γ, precedes a term of γ', then the limiting set of the sequence γ' is the first term of γ which, in γ, follows all the terms of γ', unless this limiting set is a subset of H in which case γ' is γ.

Let γ_T denote a well-ordered sequence such that (1) x is a term of γ_T if, and only if, x is the common part of $T - T \cdot H$ and some continuum which is a term of γ, (2) x precedes y in γ_T if, and only if, there exist terms x' and y' of γ such that $x = x' \cdot T$ and $y = y' \cdot T$ and x' precedes y' in γ. Let γ_H denote a well-ordered sequence described in exactly the same manner except for the substitution of γ_H for γ_T and of H for $T - T \cdot H$. Let Q, Q_T and Q_H denote the limiting sets of γ, γ_T and γ_H, respectively. It may be seen that $Q = Q_T + Q_H$. Suppose Q_T is not a subset of H. Every term of γ_T contains every one that follows it and therefore $Q_T - H \cdot Q_T$ is the common part of all the point sets of this sequence. Let P denote the first point of Q_T in the sequence α. Let ω denote a subsequence of γ such that x belongs to ω if, and only if, it contains a point that precedes P in α. The second term of γ that follows every term of ω exists but does not contain P. This involves a contradiction.

Theorem 6 does not remain true if the stipulation that, for each point P of $T - (A + B + T \cdot H)$ there is a continuum lying in H and shielding $A + B$ from P is replaced by the stipulation that for each such point P there is a continuum lying in H and intersecting every subcontinuum of M which intersects both P and $A + B$. Consider the following example.

Example 3. In a Euclidean plane, let M denote a compact indecomposable continuum and let A and B denote points belonging to different composants of M. Let H denote $A + B$. If P is a point of $M - H$, M is an irreducible continuum either from P to A or from P to B. In the first case, B intersects every subcontinuum of M that intersects both P and $A + B$. In the second case, A intersects every such continuum. But no component of H contains both A and B.

The following theorem may be easily proved with the assistance of Theorem 6.

THEOREM 7. *If the points A and B belong to different components of β, the boundary of the compact continuum M, there exists a component D of $M - \beta$ such that no component of β shields $A + B$ from D in M.*

THEOREM 8. *If, in a space satisfying Axioms 0 and 1, β is the boundary of the continuum M, D is a component of $M - \beta$ and K is a continuum intersecting D but not lying wholly in it and $M \cdot K$ is compact then every component of $K \cdot \bar{D}$ contains a point of β.*

Proof. Suppose there exists a component N of $K \cdot \bar{D}$ containing no point of β. Then there exists a subset H of $K \cdot \bar{D} \cdot (S - \beta)$ containing N such that either $N = H$ or H and $K \cdot \bar{D} - H$ are mutually exclusive and closed. There exists a domain I containing H such that $\bar{I} \cdot K \cdot \bar{D} = H$ and $\bar{I} \cdot (\beta + S - M)$ is vacuous. There exists a subcontinuum T of K lying in $\bar{I}$ and intersecting both N and $S - I$. The point set T is a connected subset of K intersecting N but not lying wholly in it. Hence it is not a subset of D. But it is a subset of $M - \beta$ and it contains a point of D. This involves a contradiction.

Theorem 8 does not remain true if the stipulation that $M \cdot K$ is compact is replaced by the stipulation that $\bar{D}$ is compact and that M is identical with K. Consider the following example.

Example 4. In a Cartesian plane E, let A and D denote the points $(0, 2)$ and $(0, 1)$ and, for each n, let B_n denote the point $(-1/n, 0)$ and let AB_n denote the straight-line interval whose end-points are A and B_n. Let both K and M denote the point set $D + AB_2 + AB_3 + \ldots$ and let S denote $M + AB_1$. Let Σ denote the subspace of E whose points are the points of S. Here β is A and the point D is a component of $M - \beta$. Though $\bar{D}$ is compact, the component D of $K \cdot \bar{D}$ does not contain A.

That Theorem 8 does not remain true if the requirement that $M \cdot K$ be compact is replaced by the requirement that space be the plane and D be the only component of $M - \beta$ may be seen from the following example.

Example 5. In a Cartesian plane E, let M denote the set of all points (x, y) such that either $-1/x^2 \leqslant y \leqslant 1/x^2$ or $x = 0$ and let K denote the set of all points (x, y) such that either $x = 0$ or $y = (2 \sin 1/x^2)/x^2$.

However, *Theorem 8 does remain true if the requirement that $M \cdot K$ be compact is replaced by the requirement that Axiom 2 hold true and $K \cdot \bar{D}$ be compact.*

THEOREM 9. *If, in a space satisfying Axioms 0 and 1, β is the boundary of the continuum M, D is a component of $M - \beta$ and K is a continuum intersecting $\bar{D}$ and such that $M \cdot K$ is compact and either $\beta \cdot \bar{D}$ is vacuous or there exists a subcontinuum of K lying in $\bar{D}$ and containing $\beta \cdot \bar{D}$, then $K \cdot \bar{D}$ is a continuum.*

Theorem 9 may be easily established with the help of Theorem 8. It does not remain true if $M \cdot K$ is replaced by $\bar{D}$. Consider the following example.

Example 6. Let space be that of Example 2, but now let K denote the point set $AB_2 + AB_3 + \ldots$, let M denote $A + (S - AB_1)$ and let D denote $S - (AB_1 + AB_2) + \ldots)$. The point set $M \cdot K$ is now K, $\bar{D}$ is compact and β, the boundary of M, is the point A which is a subcontinuum of K lying in $\bar{D}$ and coinciding with $\beta \cdot \bar{D}$. But $K \cdot \bar{D}$ is $A + O$ which is not connected.

Theorem 9 does not remain true if the stipulation that $M \cdot K$ is compact is replaced by the stipulation that $\bar{D}$ is compact even though the resulting hypothesis is strengthened by the elimination of the word "either" and "or there exists a subcontinuum of K lying in $\bar{D}$ and containing $\beta \cdot \bar{D}$." Consider the following example.

Example 7. In a Cartesian plane E, let A, A_1 and A_2 denote the points $(0, 4)$, $(0, 1)$ and $(0, 3)$, respectively and, for each n, let B_n denote $(-1/n, 0)$ and let AB_n denote the straight-line interval whose extremities are A and B_n. Let T denote a semi-circle passing through $(1, 2)$ and with its extremities at A_1 and A_2. Let K denote the point set $A_1 + A_2 + AB_2 + AB_3 + \ldots$, let M denote $K + T$ and let S denote $M + AB_1$. Let Σ denote the subspace of E whose points are the points of S. Here D is $T - (A_1 + A_2)$, β is A and $\beta \cdot \bar{D}$ is vacuous. But $K \cdot \bar{D}$ is $A_1 + A_2$ which is not connected.

THEOREM 10. *Theorem 9 remains true if the stipulation that $M \cdot K$ is compact is replaced by the stipulation that Axiom 2 holds true.*

Proof. Since Axiom 2 holds true, D is a domain whose boundary is $\beta \cdot \bar{D}$. Hence, by hypothesis, there exists a subcontinuum H of K lying in $\bar{D}$ and containing $\beta \cdot \bar{D}$. If $K \cdot \bar{D}$ is neither H nor K then $K - H$ is the sum of the point sets $(K - H) \cdot D$ and $K - K \cdot \bar{D}$. Since D is a domain these point sets are mutually separated. Hence $(K - H) \cdot D + H$ is a continuum. But this point set is $K \cdot \bar{D}$.

[1] *Amer. Math. Soc. Colloquium Pub.*, **13**, New York (1938).

CONCERNING DOMAINS WHOSE BOUNDARIES ARE COMPACT

BY R. L. MOORE

DEPARTMENT OF PURE MATHEMATICS, UNIVERSITY OF TEXAS

Communicated October 23, 1942

The following theorem may be established by an argument differing slightly from that given to prove Theorem 12 of Chapter IV of the author's "Foundations of Point Set Theory."[1]

THEOREM 1. *If, in a space satisfying Axioms 0–5, H and K are two mutually exclusive closed and compact point sets and L is a closed point set*

containing no point of H and G is a collection of regions covering H there exists a domain D containing H and such that (1) $\bar{D}$ contains no point of K and the boundary of D is a compact point set containing no point of L, (2) D is the sum of a finite number of mutually exclusive connected domains $\bar{Q}$ such that (a) every complementary domain of $\bar{Q}$ is bounded by a simple closed curve, (b) $\bar{Q}$ is the sum of a finite number of domains each bounded by a simple closed curve lying in some region of G.

THEOREM 2. *If Axioms 0–3 hold true, ω is a point and M is a closed and compact point set not containing ω and $W_1, W_2, W_3, \ldots$ is a sequence satisfying with respect to M all the requirements of Theorem 81 of Chapter I then, for each closed and compact point set T which does not intersect M, there exists a number δ such that if a closed point set K separates a point P of M from ω and lies in some region of $W\delta$ which contains P then K separates P from T.*

Proof. Suppose there exists a closed and compact point set T for which there is no such number δ. Then, for each positive integer n, there exists a region g_n belonging to W_n and containing both a Point P_n of M and a closed point set which separates P_n from ω but not from T. There exists an ascending sequence of positive integers $n_1, n_2, n_3, \ldots$ such that if, for each i, P_i' denotes the term of the sequence $P_1, P_2, P_3, \ldots$ whose subscript is n_i, then $P_1', P_2', P_3', \ldots$ converges to a point P of M. For each i, let W_i' denote the term of the sequence $W_1, W_2, W_3, \ldots$ whose subscript is n_i and let g_i' denote the term of $g_1, g_2, g_3, \ldots$ with the same subscript. By Axiom 3, the domain $S - P$ is connected. Hence, by a theorem of Dr. Harlan Cross Miller's,[2] there exists a continuum L containing $T + \omega$ and lying in $S - P$. For each n, there exists a closed point set K_n, lying in G_n', such that $S - K_n$ is the sum of two mutually separated point sets one containing ω and the other one containing both P_n' and some point of T, and therefore of L. Since L is connected it contains a point X_n belonging to K_n and therefore to g_n'. Since, for each n, X_n belongs to a region of W_n', and therefore to a region of W_n, some subsequence α of the sequence $X_1, X_2, X_3, \ldots$ converges to a point X of M. Since each point of α belongs to L therefore X belongs also to L. Hence it is distinct from P. Therefore there exists a number m such that if x, y and z are intersecting regions belonging to one of the collections $W_m, W_{m+1}, W_{m+2}, \ldots$ and x intersects y and y intersects z and x contains P then z does not contain X. There exist regions x and z belonging to W_m and containing P and X, respectively. There exists a number j, greater than m, such that P_j' belongs to x and X_j belongs to z. The region g_j' contains P_j' and X_j and lies in some region y of W_m. Since x intersects y and y intersects z, the region z does not contain X. This involves a contradiction.

The following theorem may be established with the help of Theorems 1 and 2 and Theorem 81 of Chapter I.

THEOREM 3. *If, in a space satisfying Axioms 0–5, M is a closed and com-*

pact point set there exists a sequence of domains D_1, D_2, D_3, ... such that (1) for each n, D_n contains M, and $\bar{D}_{n+1}$ is a subset of D_n and the boundary of D_n is compact, (2) for each n, D_n is the sum of a finite number of mutually exclusive connected domains Q such that every complementary domain of $\bar{Q}$ is bounded by a simple closed curve, (3) if D is a domain containing M there exists a number δ such that, for every n greater than δ, the boundary of D_n is a subset of D, (4) if D is a domain containing M and bounded by a compact point set then, for some n, D_n is a subset of D, (5) M is the common part of the domains of this sequence.[3]

AXIOM $\bar{5}$ (F. B. Jones).[4] *If A is a point of a region R and B is a point distinct from A there exists in R a closed and compact point set T separating A from B.*

THEOREM 4. *If, in a space satisfying Axioms 0–3 and $\bar{5}$, M is a closed and compact point set there exists a sequence of domains D_1, D_2, D_3, ... such that (1) for each n, $\bar{D}_{n+1}$ is a subset of D_n, (2) for each m, the boundary of D_n is compact, (3) if D is a domain containing M there exists a number δ such that, for every n greater than δ, the boundary of D_n is a subset of D, (4) if D is a domain containing M and bounded by a compact point set then, for some n, D_n is a subset of D, (5) M is the common part of the domains of this sequence.*

Theorem 4 may be established with the aid of Theorem 81 of Chapter I, Theorem 2, and a theorem corresponding to Theorem 1.

THEOREM 5. *In a space satisfying Axioms 0–3 and $\bar{5}$, no domain whose boundary is compact has uncountably many components.*

Proof. Suppose there exists a domain D bounded by a compact point set M and having uncountably many components. There exists a sequence of domains D_1, D_2, D_3, ... satisfying, with respect to M, all the conditions of Theorem 4. If x is a component of D there exists a positive integer n_x such that $\bar{D}_{n_x}$ does not contain x. There exists a positive integer k such that, for uncountably many components x of D, n_x is k. For each such component x of D, the common part of x and β_k, the boundary of D_k, exists and is a domain with respect to β_k. But since β_k is compact there do not exist uncountably many mutually exclusive domains with respect to it. This involves a contradiction.

It is not true that if Axioms 0, 1, 2 and $\bar{5}$ hold true then, for each point P, there exists a sequence of domains D_1, D_2, D_3, ..., each having a compact boundary, such that P is their common part or that no domain with a compact boundary has uncountably many components. Consider the following example.

Example 1. Suppose E is a Cartesian space of three dimensions. Interpret the word point to mean a point of E and call a point set M a region if and only if either M is the interior of a sphere with center at O or M is a segment, not containing O, of some straight line through O. There exists no countable collection of domains with compact boundaries such that O

is their common part. Furthermore the set of all points distinct from O is a domain whose boundary is O and this domain has uncountably many components.

THEOREM 6. *If, in a space satisfying Axioms 0, 1 and 2, the common part of the continuum K and the closed point set M is the sum of the two mutually exclusive closed point sets H and L then there exists an arc having in common with M only its end-points which belong to H and L, respectively.*

Proof. For each n, let H_n denote the collection of all connected domains D such that (1) D is a subset of some region of G_n[5] and (2) D either intersects $M \cdot K$ or does not intersect M. There exists a simple chain C_1 of domains of H_1 such that (1) M intersects no link of C_1 except the first one and the last one, (2) the first link of C_1 intersects H and the last one intersects L. There exists[6] a simple chain C_2 of domains of H_2 such that (1) the closure of each link of C_2 is a subset of some link of C_1, (2) if a link x of C_2 lies in a link y of C_1 then every link that follows x in C_2 lies either in y or in some link that follows y in C_1, (3) M intersects no link of C_2 except the first one and the last one, and (4) the first one and last one intersect H and L, respectively. This process may be continued. Thus there exists an infinite sequence of simple chains $C_1, C_2, C_3, \ldots$ such that (1) each link of C_n belongs to H_n, (2) the closure of each link of C_{n+1} is a subset of some link of C_n, (3) if a link x of C_{n+1} lies in a link y of C_n then each link that follows x in C_{n+1} lies either in y or in some link that follows y in C_n, (4) the first link of C_n intersects H and the last one intersects L and no other one intersects M. For each n, let C_n^* denote the sum of all the links of C_n. Let C denote the common part of the point sets $C_1^*, C_2^*, C_3^*, \ldots$. For each n, let α_n and β_n denote the first and last links, respectively, of the chain C_n. For each n, α_n contains $\bar{\alpha}_{n+1}$ and α_n is a subset of some region of G_n. Hence the point sets $\alpha_1, \alpha_2, \alpha_3, \ldots$ have only one point in common. Call this point A. Let B denote the point common to the point sets $\beta_1, \beta_2, \beta_3, \ldots$. The points A and B belong to H and L, respectively, and, for each n, C_n is a simple chain from A to B. The point set C is[6] an arc from A to B. Clearly $C \cdot M = A + B$.

THEOREM 7. *If, in a space satisfying Axioms 0, 1 and 2, M is a continuum such that every component of $S - M$ has a connected boundary and K is a continuum containing the boundary of M then $M \cdot K$ is a continuum.*

Proof. Suppose $M \cdot K$ is the sum of two mutually exclusive closed point sets H and L. By Theorem 6 there is a component D of $S - M$ whose boundary γ intersects both H and L. The point sets $\gamma \cdot H$ and $\gamma \cdot L$ are subsets of γ, and γ is a connected subset of $M \cdot K$. Thus there is a component of $M \cdot K$ intersecting both H and L. This involves a contradiction.

THEOREM 8. *If, in a space satisfying Axioms 0, 1 and 2, M is a closed point set with a compact boundary β then there exists a closed and compact point set M' lying in M, containing β and such that if D is a component of*

$M - \beta$ such that every component of $S - \bar{D}$ has a connected boundary then $M' \cdot \bar{D}$ is a continuum containing the boundary of D.

Proof. There exists[2] a compact continuum N containing β. Let M' denote $M \cdot N$. Suppose D is a component of $M - \beta$ such that every component of $S - \bar{D}$ has a connected boundary. The boundary of D is a subset of M' since it is a subset of β. By Theorem 7, $M' \cdot D$ is a continuum.

THEOREM 9. *If, in a space satisfying Axioms 0, 1 and 2, O is a point of a continuum M whose boundary β is compact and every component of the complement of M has a connected boundary, but M is not locally connected at O then there exist a domain D containing O and a sequence $M_1, M_2, M_3, \ldots$ of mutually exclusive subcontinua of M such that (1) for each n, M_n is a component of $M \cdot \bar{D}$ containing a point of D and a point of the boundary of D, (2) the sequence $M_1, M_2, M_3, \ldots$ converges to a subcontinuum of M containing O and a point of the boundary of D.*

Proof. There exist a domain D containing O and a sequence of distinct points $P_1, P_2, P_3, \ldots$ belonging to $M \cdot D$ such that (1) this sequence converges to O and (2) no component of $M \cdot \bar{D}$ contains more than one of the points $O, P_1, P_2, P_3, \ldots$. There exists a compact continuum T containing $\beta + O + P_1 + P_2 + P_3 + \ldots$. Let L denote $T \cdot M$. By hypothesis and Theorem 7, L is connected. For each n, L contains P_n and O but no connected subset of D containing P_n and O. Hence it contains a point of $S - D$. But L is compact. Hence L_n, the component of $L \cdot \bar{D}$ that contains P_n, contains a point of the boundary of D. The point sets $L_1, L_2, L_3, \ldots$ are mutually exclusive subsets of the closed and compact point set $L \cdot \bar{D}$. It follows that there exists an increasing sequence of positive integers $n_1, n_2, n_3, \ldots$ such that $L_{n_1}, L_{n_2}, L_{n_3}, \ldots$ converges to a compact continuum containing O and lying in $L \cdot \bar{D}$ and intersecting the boundary of D.

Theorem 9 does not remain true on the omission from its hypothesis of the condition requiring that Axiom 2 be satisfied. Consider the following example.

Example 2. In a Cartesian plane E, let O, A, B, C, D and E denote the points $(0, 0)$, $(0, -1)$, $(1, -1)$, $(2, -1)$, $(2, 0)$ and $(1, -2)$, respectively, and let OA, AC, CD and BE denote straight line intervals with end-points as indicated. Let K denote the set of all points of the graph of $y = (1/x) \sin (1/x)$ whose abscissae lie between 0 and 2, let M denote the point set $K + OA + AC + CD$ and let S denote $M + BE$. Let Σ denote the subspace of E whose points are the points of S. In this space the continuum M is not locally connected at O. Its boundary consists of the single point B and the only component of $S - M$ has B as its boundary. But M has no continuum of condensation.

Theorem 9 does not remain true on the omission of the requirement that every component of the complement of M have a connected boundary.

This may be seen with the aid of Example 1 of my paper "Concerning Accessibility."[3]

THEOREM 10.[7] *If, in a space satisfying Axioms 0, 1 and 2, M is a continuous curve and N is a subcontinuum of M whose boundary with respect to M has no continuum of condensation and every component of $S - N$ has a connected boundary then N is a continuous curve.*

Theorem 10 may be established by an argument identical with that given in lines 3 to 27 of page 114, in Chapter II, except for the substitution of M' for N in line 24 and of "by Theorem 9 of the present paper" in place of "with the aid of Theorem 22, by a modification of the argument employed to prove Theorem 8."

THEOREM 11. *If, in a space satisfying Axioms 0–5, M is a continuum whose boundary is compact then M is compactly connected.*

Proof. The boundary of M is a subset of a compact continuum K. Since the boundary of every component of $S - M$ is connected therefore, by Theorem 7, $M \cdot K$ is a continuum. That M is compactly connected follows as in the proof of Theorem 10 of "Concerning Accessibility."

THEOREM 12. *If, in a space satisfying Axioms 0–5, M is a continuous curve and N is a subcontinuum of M whose boundary has no continuum of condensation then N is a continuous curve.*

Theorem 12 may be easily established with the aid of Theorem 10 and the fact that, in a space satisfying Axioms 0–5, every compact boundary of a complementary domain of a continuum is itself a continuum.

In the hypotheses of some of the numbered theorems of Chapter IV it is stipulated that certain domains form a semi-contracting set. With the aid of the results obtained in this paper it may be seen that all of these theorems remain true on the omission of this stipulation.

[1] *Am. Math. Soc. Colloquium Pub.*, Vol. XIII, New York (1932). Theorem 10 may be strengthened in a similar manner by the interpolation of "and L is a closed point set containing no point of M" between "M" and "there" and the substitution, at the end, of "L or of M" in place of "M." That Theorem 13 remains true on the removal of the requirement that K be compact may be shown by an argument identical with that given in Chapter IV (to prove this theorem in its original form) except for the substitution of the following two sentences, in place of the second sentence of that argument: "By Theorem 41 of Chapter I, there exists a domain Q containing H but no point of $K + T$ and such that $\bar{Q} - Q$ contains no point of $M + K + T$. Let L denote $M \cdot Q$ and let N denote $(M + K) - L$. The point set $M + K + T$ is the sum of the two point sets L and $N + T$ which contain H and $K + T$, respectively." Except in the case of Axiom $\bar{5}$, numbered axioms and chapters herein referred to are axioms and chapters of this book. The letter S denotes the set of all points.

[2] Miller, H. C., "A Theorem Concerning Closed and Compact Point Sets Which Lie in Connected Domains," *Bull. Am. Math. Soc.*, **46**, 520–521 (1940).

[3] It does not follow even for the case where M is a single point, that if D_n is a domain containing M then, for some n, D_n is a subset of D. Cf. Example 2 of my paper "Concerning Accessibility," these PROCEEDINGS, **25**, 648–653 (1939). In the space of this

example there does not exist a sequence of domains $D_1, D_2, D_3, \ldots$ closing down on the point O and such that, for each n, the boundary of D_n is compact.

[4] Jones, F. B., "Concerning Certain Topologically Flat Spaces," *Trans. Am. Math. Soc.*, 42, 53–93 (1937).

[5] See Axiom 1.

[6] See the proof of Theorem 1 of Chapter I.

[7] See the proposition labeled "Theorem 25" in Chapter II. That this proposition is not a consequence of Axioms 0, 1 and 2 may be seen with the aid of Example 1.

ON SETS OF INTEGERS WHICH CONTAIN NO THREE TERMS IN ARITHMETICAL PROGRESSION

BY R. SALEM AND D. C. SPENCER

DEPARTMENTS OF MATHEMATICS, HARVARD UNIVERSITY AND MASSACHUSETTS INSTITUTE OF TECHNOLOGY, AND STANFORD UNIVERSITY

Communicated October 26, 1942

Let S be a set of non-negative integers, all different from one another. We say briefly that S is "progression-free" if any three distinct integers of S never form an arithmetical progression, i.e., if $a + a' \neq 2a''$ whenever a, a', a'' are different and belong to S.

If the elements of a progression-free set S do not exceed a given N, then the number of elements of S has clearly a maximum $\nu = \nu(N)$.

It has been widely conjectured that, as $N \to \infty$, $\nu(N) = 0(N)^{\alpha}$ where α is a positive constant inferior to 1. A more precise conjecture has assigned to α the value $\log 2/\log 3$ which corresponds to the progression-free sequence of integers whose digits in the ternary scale are 0 and 2 only.[1]

The purpose of the present note is to prove that the conjecture $\nu(N) = 0(N^{\alpha})$ is false for every $\alpha < 1$. We shall prove that, as $N \to \infty$,

$$\nu(N) > N^{1 - \frac{\log 2 + \epsilon}{\log \log N}}$$

for every $\epsilon > 0$.

Let d be an integer >2 and n an integer divisible by d. Having fixed d and n, let $S(d, n)$ be the set of all integers given by the expression

$$A = a_1 + a_2(2d - 1) + \ldots + a_n(2d - 1)^{n-1}$$

where the "digits" a_i are integers subjected to the following condition: exactly n/d digits are equal to zero, n/d digits are equal to 1, n/d digits are equal to 2, etc...., and n/d digits are equal to $d - 1$. Thus the number of integers of $S(d, n)$, all different, is

$$\mu(d, n) = \frac{n!}{[(n/d)!]^d} \tag{1}$$

On the other hand, for all numbers A of $S(d, n)$ we have

$$A < (2d - 1)^n. \tag{2}$$

The set $S(d, n)$ is progression-free. Suppose that A, A', A'' belong to the set and that $A + A' = 2A''$. Let a_i, a_i', a_i'' be the digits of rank i in A, A', A'', respectively. Since $a_i + a_i' \leqslant 2d - 2$ and $2a_i'' \leqslant 2d - 2$, the equality $A + A' = 2A''$ implies $a_i + a_i' = 2a_i''$ for all i. Now there are in A'' exactly n/d digits equal to zero, and if $a_h'' = 0$, then necessarily $a_h = a_h' = 0$; i.e., the n/d digits equal to zero occupy the same places in A, A' and A''. Next, there are, in A'', n/d digits equal to 1, and if $a_k'' = 1$, then since $a_k \neq 0$ and $a_k' \neq 0$, the equality $a_k + a_k' = 2a_k''$ implies $a_k = a_k' = 1$; i.e., the n/d digits equal to 1 correspond in A, A', A''. Generally, if $a_l'' = m$, a_l and a_l' being different from $0, 1, 2, \ldots m - 1$, then $a_l = a_l' = m$ and the n/d digits equal to m have the same ranks in A, A', A''. Going up to $m = d - 1$, we prove that $A = A' = A''$, that is to say $S(d, n)$ is progression-free.

Now if n and n/d are large enough we have by (I)

$$\mu(d, n) > \frac{n^n\sqrt{2\pi n}\ e^{-n}}{[(n/d)^{n/d}\sqrt{2\pi(n/d)}\ e^{-n/d}]^d}\frac{1}{C^d}$$

C being a constant (as near to 1 as we please). Thus

$$\mu(d, n) > (d/\gamma n)^{d/2} d^n \tag{3}$$

γ being a constant (as near to 2π as we please).

Let us now fix an N and let us choose d such that

$$(2d - 1)^{d\omega(d)} \leqq N < (2d + 1)^{(d+1)\omega(d+1)} \tag{4}$$

where $\omega(d)$ is an integer increasing infinitely with d and such that $\dfrac{\omega(d)}{\log d} \to \infty$ but $\dfrac{\log \omega(d)}{\log d} \to 0$ as $d \to \infty$. Let us construct the set $S(d, n)$ with $n = d\omega(d)$. We have by (2), (3) and (4)

$$\nu(N) \geqslant \nu[(2d - 1)^{d\omega(d)}] \geqslant \mu(d, d\omega(d)) > \left(\frac{1}{\gamma\omega(d)}\right)^{d/2} d^{d\omega(d)}$$

$$\frac{\nu(N)}{N} > \left(\frac{1}{\gamma\omega(d)}\right)^{d/2} \frac{d^{d\omega(d)}}{(2d + 1)^{(d+1)\omega(d+1)}}$$

Now, as $N \to \infty$, $d \to \infty$, and

$$\log\left(\frac{N}{\nu(N)}\right) < (d+1)\omega(d+1)\log(2d+1) - d\omega(d)\log d +$$

$$\frac{d}{2}\log\omega(d) + \frac{d}{2}\log\gamma = d\omega(d)[\log 2 + o(1)], \quad (5)$$

if we suppose, as we may, that $\omega(d)$ increases regularly enough to have $\omega(d+1) - \omega(d) = o(1)$. By (4)

$$\log N \geqslant d\omega(d)\log(2d-1)$$

$$\log\log N < \log(d+1) + \log\omega(d+1) + \log\log(2d+1)$$

and so

$$\frac{\log N}{\log\log N} > d\omega(d)[1 + o(1)]. \quad (6)$$

From (5) and (6) it plainly follows that, as $N \to \infty$

$$\nu(N) > N^{1 - \frac{\log 2 + \epsilon}{\log\log N}}$$

for every $\epsilon > 0$.

Remark.—The sequence constructed above is finite and the construction depends on N. Therefore it should be pointed out that by a slight modification of the argument, we can form an infinite "progression-free" sequence of integers such that the number of terms of the sequence not exceeding N is, for $N \to \infty$, greater than $N^{1 - \frac{a}{\log\log N}}$, a being a constant.

Extension to Sets of Points.—Let E be a set of points in (0, 1) such that if x and y belong to E, then $(x+y)/2$ belongs to E if and only if $x = y$ (property P). It is known that E is of measure zero.[2] An adaptation of the above argument yields a perfect set E having the property P and whose Hausdorff dimensionality is greater than every $\alpha < 1$. The proof, together with other remarks on sets of points having the property P, will appear elsewhere.

[1] See Erdos and Turan, *Jour. London Math. Soc.*, **11**, 261–264 (1936).

[2] See Ruziewicz, S., *Fundamenta Mathematicae* **7**, 141–143 (1925).

SOME ADVANCES IN THE COMBINATORY THEORY OF QUANTIFICATION

BY HASKELL B. CURRY

DEPARTMENT OF MATHEMATICS, THE PENNSYLVANIA STATE COLLEGE

Communicated October 17, 1942

The purpose of this paper is to report the present status of certain researches which have been interrupted by the war. These depend on a series of previous papers which will be referred to by abbreviations listed in a footnote.[1] The researches are those sketched in CFM, §§ 6–8. The notations of CFM and CCT will be adhered to, except as later specified.

1. Preliminary conventions; the system $\mathfrak{F}_0$. The basis of the investigation is that form of λ-formalism which has been proved equivalent to the system $\mathfrak{L}_2$ (CCT, pp. 57 ff.). To this are adjoined a unary predicate, expressed as usual by prefixed '$\vdash$', and an unspecified class of *canonical terms*. The rudimentary system so formed, with the rules stated below, will be called $\mathfrak{F}_0$.

Unspecified terms of $\mathfrak{F}_0$ and its extensions will be denoted by capital German letters; unspecified canonical terms by small Greek letters. When the statement of a postulate, theorem, rule, etc., involves such letters it is understood that arbitrary substitutions of appropriate terms can be made. The symbols '$\rightarrow$', '&', '$\equiv$' will be used as in PKR, § 2.4. The symbols '$=$', '$\geqq$' will be used, respectively, for 'is convertible to' and 'is reducible to', both in the sense of CCT, § 2.

The rules for $\mathfrak{F}_0$ (and its extensions) are: (1) if $\alpha = \mathfrak{B}$, then $\mathfrak{B}$ is canonical, and (2) the rule

RULE CONV.: $\vdash \mathfrak{A} \,\&\, \mathfrak{A} = \mathfrak{B} \rightarrow \vdash \mathfrak{B}$.

2. The system F_1. Suppose we adjoin to $\mathfrak{F}_0$: (1) the primitive term F, (2) the rule given for F in CFM, which will be called RULE F, (3) the rule that $F\alpha\beta$ is always canonical, and (4) Postulates (*FK*) and (*FS*) of CFM. The system so formed will be called $\mathfrak{F}_1$.

In $\mathfrak{F}_1$ all the theorems of FPF, §§ 2, 3, 4 are valid if appropriate changes (mostly replacing quantified variables by Greek letters and taking implications metatheoretically) are made. However, the proof of such theorems as 3.11 and 3.12 is facilitated by the following.

THEOREM. *If* $\mathfrak{X}$ *is a combination of the variables* $x_1, \ldots, x_n$ *and the terms* $\mathfrak{A}_1, \ldots, \mathfrak{A}_m$, *and if* $\alpha_1, \ldots, \alpha_m$ *are such that*

$$\vdash \alpha_i \mathfrak{A}_i \qquad i = 1, 2, \ldots, m; \tag{1}$$

if, further, it follows from the assumptions

$$\vdash \xi x_j \qquad j = 1, 2, \ldots, n$$

and the theorems (1), by following out the construction of $\mathfrak{X}$ *and applying Rule F, that*

$$\vdash \eta \mathfrak{X};$$

then without hypothesis

$$\vdash F_n \xi_1 \ldots \xi_n \eta (\lambda^n x_1 \ldots x \mathfrak{X}_n).$$

This theorem can be proved directly by induction with greater ease than most of the theorems of FPF, § 3.

The consistency of $\mathfrak{F}_1$ follows readily provided only that canonical terms are such that $\alpha\mathfrak{X}$ can reduce only to terms of the form $\alpha'\mathfrak{X}'$ where $\alpha \geqq \alpha'$ and $\mathfrak{X} \geqq \mathfrak{X}'$.[2]

3. The system $\mathfrak{F}_2$. To form the system $\mathfrak{F}_2$ adjoin Ξ to $\mathfrak{F}_0$ as new primitive term and also the rule, henceforth called RULE Ξ, given for Ξ in CFM. We also need the rule that $\Xi\alpha\beta$, $\alpha\mathfrak{X}$ and $\lambda x\alpha$, are canonical.[3] Next define

$$\mathfrak{A} \supset_x \mathfrak{B} \equiv \Xi(\lambda x \mathfrak{A})(\lambda x \mathfrak{B}).$$

The postulates are then:

POST. (ΞK). $\vdash \alpha x \supset_x \cdot \beta x y \supset_y \alpha x$

POST. (ΞS). $\vdash \alpha u \supset_u \cdot \beta u v \supset_v \gamma x u v \cdot \supset_x : \alpha w \supset_w \beta_w(yw) \cdot \supset_y \cdot \alpha z \supset_z \gamma x z(yz)$.

If F is defined by (10) of CFM, this system includes $\mathfrak{F}_1$. The following deduction theorem can also be proved:

THEOREM. *If from the assumption that*

$$\vdash \xi x,$$

together with certain axioms which do not contain x, *it is derivable by the rules of* $\mathfrak{F}_2$ *that*

$$\vdash \mathfrak{X};$$

then from these same axioms it is derivable absolutely within $\mathfrak{F}_2$ *that*

$$\vdash \xi x \supset_x \mathfrak{X},$$

provided that the axioms and the terms corresponding to $\mathfrak{A}$, $\mathfrak{B}$ *in every application of Rule* Ξ *are canonical.*

The consistency theorem of § 5 removes the last proviso. On the other hand, the theorem, apparently, cannot be iterated to take care of multiple premises such as $\vdash \xi x$, $\vdash \eta x y$; for when we eliminate one premise we introduce new axioms, derived from the postulates of $\mathfrak{F}_2$, which may contain other variables. This shows that the postulates for $\mathfrak{F}_2$ are still in a tentative state.

4. The system $\mathfrak{F}_3$. The system obtained by adjoining to $\mathfrak{F}_0$ the terms II and P with the rules set down for them in CFM will be called $\mathfrak{F}_3$. For canonical terms we must have $\Pi\alpha$, $P\alpha\beta$, $\alpha\mathfrak{X}$ and $\lambda x\alpha$.[4] The postulates will be some as yet undetermined set sufficiently strong to include $\mathfrak{F}_2$ (with Ξ defined as in (8) of CFM) and the postulates stated in CFM.

5. Combinatory verifiability; the systems $\mathfrak{V}$. For the study of the consistency of $\mathfrak{F}_2$ and $\mathfrak{F}_3$ it is expedient to introduce systems $\mathfrak{V}_2$ and $\mathfrak{V}_3$ which have the same relation to the former that the Gentzen system LJ has to the Heyting calculus. These involve two predicates of an undetermined number of arguments, viz.,

$$\text{Can } (\mathfrak{M}_1, \ldots, \mathfrak{M}_n, \mathfrak{A}) \qquad n \geqq 0 \tag{1}$$

$$\text{Ver } (\mathfrak{M}_1, \ldots, \mathfrak{M}_n, \mathfrak{A}) \qquad n \geqq 0. \tag{2}$$

The interpretations of the formulas (1), (2) are "$\mathfrak{A}$ is canonical (or verifiable, respectively) on the basis $\mathfrak{M}_1, \ldots, \mathfrak{M}_n$"; when $n = 0$ this will mean simply that the term $\mathfrak{A}$ is canonical or verifiable. The formula

$$\text{Reg } (\mathfrak{M}_1, \ldots, \mathfrak{M}_n, \mathfrak{A}) \qquad n \geqq 0. \tag{3}$$

will be used to stand for either (1) or (2). A sequence of terms to which (1), (2) or (3) applies will be called a canonical, verifiable or regular sequence, respectively; in such a sequence $\mathfrak{M}_1, \ldots, \mathfrak{M}_n$ are the antecedents and $\mathfrak{A}$ the consequent.

In the following statement of the rules '$\binom{\mathfrak{U}}{x}\mathfrak{A}$' means the result of substituting $\mathfrak{U}$ for x in $\mathfrak{A}$; the accents should be ignored for the present.

I. $\text{Can } (\mathfrak{M}_1, \ldots, \mathfrak{M}_n, \mathfrak{A}') \rightarrow \text{Ver } (\mathfrak{M}_1, \ldots, \mathfrak{M}_n, \mathfrak{A}, \mathfrak{A})$.

II. *If* $k_1, \ldots, k_r$ *is any sequence such that each* k_j *has one of the values* $1, 2, \ldots, n$, *then*

$$\text{Can } (\mathfrak{M}_1, \ldots, \mathfrak{M}_{n-1}, \mathfrak{M}_n') \,\&\, \text{Reg } (\mathfrak{M}_{k_1}, \ldots, \mathfrak{M}_{k_r}, \mathfrak{A}) \rightarrow \text{Reg } (\mathfrak{M}_1, \ldots, \mathfrak{M}_n, \mathfrak{A}).$$

III. *If* $\mathfrak{A} \geqq \mathfrak{B}$, $\mathfrak{M}_k \geqq \mathfrak{N}_k$ *then*

$$\text{Reg } (\mathfrak{N}_1, \ldots, \mathfrak{N}_n, \mathfrak{B}) \rightarrow \text{Reg } (\mathfrak{M}_1, \ldots, \mathfrak{M}_n, \mathfrak{A}).$$

IV 1. *If* x *does not occur in* $\mathfrak{M}_1, \ldots, \mathfrak{M}_n$, *then*

$$\text{Reg } (\mathfrak{M}_1, \ldots, \mathfrak{M}_n, \mathfrak{A}, \mathfrak{B}) \rightarrow \text{Reg } (\mathfrak{M}_1, \ldots, \mathfrak{M}_n, \mathfrak{A} \supset_x \mathfrak{B}).$$

IV 2. *If* x *does not occur in* $\mathfrak{M}_1, \ldots, \mathfrak{M}_m$, *then*

$$\text{Can } (\mathfrak{M}_1, \ldots, \mathfrak{M}_m, \mathfrak{A}, \mathfrak{B}') \,\&\, \text{Ver } (\mathfrak{M}_1, \ldots, \mathfrak{M}_m, \tbinom{\mathfrak{U}}{x}\mathfrak{A}) \,\&\, \text{Reg } (\mathfrak{M}_1, \ldots, \mathfrak{M}_m, \tbinom{\mathfrak{U}}{x}\mathfrak{B}, \mathfrak{N}_1, \ldots, \mathfrak{N}_n, \mathfrak{C}) \rightarrow \text{Reg } (\mathfrak{M}_1, \ldots, \mathfrak{M}_n, \mathfrak{A} \supset_x \mathfrak{B}, \mathfrak{N}_1, \ldots, \mathfrak{N}_n, \mathfrak{C}).$$

These are the rules for $\mathfrak{V}_2$; for $\mathfrak{V}_3$ IV should be replaced by analogous rules for II and P.

On this basis it can be shown that: (1) if a sequence is canonical, so is the subsequence obtained by omitting the consequent; (2) every verifiable sequence is canonical; (3) arbitrary substitutions can be made for the variables; (4) a theorem roughly the converse of IV holds; and (5) there is a theorem, called the *elimination theorem*, which is analogous to Gentzen's "Schnitt"—in particular it allows an antecedent to be dropped when it is verifiable on the basis of its predecessors. Also (6) canonicalness is invariant with respect to conversion.

These theorems presuppose that there are certain elementary canonical terms beginning with constants other than S, K, for which terms there are specifications consistent with the theorems. An example of such a specification is: Can $(Q\mathfrak{X}\mathfrak{Y})$, Ver $(Q\mathfrak{X}\mathfrak{X})$. Every term has a certain finite order, with reference to constructions by means of Ξ from these elementary constituents; the proof of the elimination theorem involves an induction with respect to this order.

In applying this argument to the consistency of $\mathfrak{F}_2$ and $\mathfrak{F}_3$ certain changes in the definitions of canonicalness for these systems have to be made.[5] When this is done we have:

THEOREM. *If the axioms of the systems $\mathfrak{F}_2$ and $\mathfrak{F}_3$ are verifiable, then all the theorems are verifiable.*

Since S, K, Ξ, Π, etc., are not canonical and QSK (on the above assumption) is canonical but not verifiable, the consistency of $\mathfrak{F}_2$ and $\mathfrak{F}_3$ is proved.

6. The systems $\mathfrak{F}^*$. We consider now the problem of expressing the "postulates" of the systems $\mathfrak{F}$, which are axiom schemes, as single axioms. There are two ways of doing this: (1) by introducing quantifiers of higher order; (2) by postulating a term H, representing canonicalness, and generalizing with the quantifiers already present. The former course is not investigated here at all; the second only partially. Certain remarks concerning it are as follows.

First there must be a notion of relative canonicalness. It is not sufficient to have $\Xi\mathfrak{A}\mathfrak{B}$ canonical when $\mathfrak{A}$ and $\mathfrak{B}$ are; there must be circumstances under which $\mathfrak{A}$ is verifiable only for certain values of x, and $\mathfrak{B}$ canonical for the same values.

Secondly, one cannot postulate $H(H\mathfrak{X})$. For on the assumption as to relative canonicalness made below, there would then be, for any canonical $\mathfrak{B}$, a canonical term $\mathfrak{A}$ such that $\mathfrak{A} = H\mathfrak{A}\supset \cdot \mathfrak{A}\supset\mathfrak{B}$, and from this a contradiction arises much as in IFL.[6] The system is inconsistent if $\vdash H^n\mathfrak{X}$ is postulated for any specific n.[7]

Thirdly, there seems to be no use in considering $\mathfrak{F}_3^*$. For if Ξ is expressible only through Π and P, then from $\vdash \Xi H\mathfrak{A}$ we must infer $\vdash P(H\mathfrak{X})(\mathfrak{A}\mathfrak{X})$ for any $\mathfrak{X}$, even when neither $H\mathfrak{X}$ nor $\mathfrak{A}\mathfrak{X}$ is canonical. It is

difficult to see how we can get a verifiability theorem in such a case. I suspect that $\mathfrak{F}_3^*$ (on any reasonable formulation) is inconsistent.

On the other hand, it is probably possible to formulate the system $\mathfrak{F}_2^*$ and to prove its consistency by constructing an appropriate $\mathfrak{B}^*$ (see § 7). Moreover, this can presumably be done so as to remove the difficulties in regard to $\mathfrak{F}_2$ mentioned in § 3. For conjectures as to the possible significance of this for mathematics see CFM, § 8.

7. The system $\mathfrak{B}_2^*$. The formulation of the system $\mathfrak{B}_2^*$ will not be given here in detail. However, the changes which must be made in the formulation of § 5 are essentially as follows:

1. Care must be used with variables, since now substitutions for variables cannot be made except when certain premises are fulfilled. We must distinguish three kinds of variables. A variable 'x' occurring in a specific place in a term of a sequence shall be said to be *apparent* there if that place is part of the scope of a prefix 'λx', otherwise it is *real;* if it is real but is also real in some previous term of the sequence it shall be said to be *bound;* otherwise it is *free.* Substitutions are not permitted for any of these—such substitutions are taken care of by the elimination theorem. It may be desirable to have a fourth class of variables, called *indeterminates*, for which substitutions can be made. It is advisable to use different classes of letters for indeterminates and apparent variables than those which are used for real variables. Some changes to this effect should be made in IV.

2. Where the consequents of certain sequences in § 5 are primed it is to be understood that there are values for the free variables for which the formula holds. These values may be functions of the bound variables and indeterminates.

3. The suggested rules for H are

V 1. Can $(\mathfrak{M}_1, \ldots, \mathfrak{M}_n, \mathfrak{A}) \rightarrow$ Reg $(\mathfrak{M}_1, \ldots, \mathfrak{M}_n, H\mathfrak{A})$.

V 2. *If* (*a*) Can $(\mathfrak{M}_1, \ldots, \mathfrak{M}_m, \mathfrak{A}')$ (1)

and if (*b*) *from* Can $(\mathfrak{M}_1, \ldots, \mathfrak{M}_m, \mathfrak{A})$ (2)

it is deducible[8] *that*

$$\text{Reg } (\mathfrak{N}_1, \ldots, \mathfrak{N}_n, \mathfrak{B}) \tag{3}$$

then

$$\text{Reg } (\mathfrak{M}_1, \ldots, \mathfrak{M}_m, H\mathfrak{A}, \mathfrak{N}_1, \ldots, \mathfrak{N}_n, \mathfrak{B}). \tag{4}$$

4. The form of the rule V requires reformulation of the concept of a proof. If we call an ordinary proof a deduction—which is a chain of formulas connected according to the rules—we have here to do with a chain of deductions, where a whole deduction of lower order can be premise

for a formula in one of higher order. The exact formulation of this requires considerable space.

5. The induction used in the proof of the elimination theorem will be of a more complicated nature.

Can the elimination theorem be proved from these premises, or some suitable modifications of them, by a finite or transfinite induction? On that question research is unfinished.

[1] These abbreviations consist of the initials of the first three principal words of the title. The citations are

CCT. *Jour. Symbolic Logic*, **6**, 54–61 (1941).
CFM. *Jour. Symbolic Logic*, **7**, 49–64 (1942).
FPF. *Tôhoku Math. Jour.*, **41**, 371–401 (1936).
IFL. "*Jour. Symbolic Logic*, **7**, 115–117 (1942).
PKR. *Trans. Amer. Math.* **50**, 454–516 (1941).

[2] For it can be shown that if $\vdash \mathfrak{x}$, then $\mathfrak{x}$ reduces to a term having that relation to a formula of the algebra of pure implication which was noted in CFM.

[3] This is an over-simplification. In $\mathfrak{F}_{\lambda\beta}$ it would be more appropriate to have different ranks of canonicalness with the rules: (1) If α and β are canonical of rank 1, then $\Xi\alpha\beta$ is canonical of rank 0; (2) if α is canonical of rank n then $\lambda x\alpha$ is canonical of rank $n + 1$ and, if $n > 0$, $\alpha\mathfrak{x}$ is canonical of rank $n - 1$. Cf. end of § 5.

[4] Cf. the preceding footnote.

[5] Cf. footnote [3].

[6] One can, of course, introduce notions of canonicalness of higher order. For conjectures relative to this see CFM, § 8.

[7] Note H^n here has the significance given to the nth power by Rosser, *Annals of Mathematics*, **36**, 129 (1935).

[8] The free variables of (2) being carried as indeterminates.

INDEX

Aberrations, chromosomal (Sax), 229.
Achlya, sexual hormones in (Raper), 509.
Advances in combinatory theory of quantification (Curry), 564.
Algal nomenclature (Papenfuss), 446.
Algebraic moduls (Harrison), 410.
Algebras, homogeneous measure (Maharam), 108.
Aminobenzoicless mutant (Tatum and Beadle), 234.
Amphibian development (Child), 339.
Amplitudes, photographic (Shapley), 501.
Analytical theory of semi-groups (Hille), 421.
Anaphase movement of chromosomes (Rhoades and Vilkomerson), 433.
ANDERS, M. V. See Sawin, P. B., 123.
ANDERSON, T. F. See Luria, S. E., 127.
Anticomplimentary action, neutralization of (Tyler), 391.
Antifertilizin (Tyler), 391.
Association of three attributes (Wilson and Worcester), 384.
Asymptotic relations (Bateman), 371.
Ataxia (Sawin, Anders and Johnson), 123.
Atoms, metallic (Lukesh), 277.
Attributes, association of (Wilson and Worcester), 384.
Automorphisms (Miller), 368.
Autosomal recessive factor (Fano), 119.
Auxin (Wildman and Gordon), 217.
Avian egg (Romanoff and Bless), 306.
Azotobacter (Wilson, Burris and Lind), 243.

Bacteria, crown-gall (Braun and Laskaris), 468.
Bacteria, intestinal (Burkholder and McVeigh), 285.
Bacteriophages, identification and characterization of (Luria and Anderson), 127.
BARTLETT, N. R., AND HUDSON, G. E. Theory of the Effects of Light Intensity and Duration in Determining Visual Responses, 289.
BATEMAN, H. Some Asymptotic Relations, 371.
——. An Orthogonal Property of the Hypergeometric Polynomial, 374.
BEADLE, G. W. See Tatum, E. L., 234.
Bernoulli numbers (Vandiver), 324.
Binary systems (Kopal), 133.
Biochemical reactions (Tatum and Beadle), 234.
Bioelectric potentials of avian egg (Romanoff and Bless), 306.
Biorthogonal expansions (Tseng), 170.
BLESS, A. A. See Romanoff, A. L., 306.
BOAS, R. P., JR. Inversion of a Generalized Laplace Integral, 21.
Bodies, more than three (Garcia), 425, 428.
BONNER, J. A Reversible Growth Inhibition of Isolated Tomato Roots, 321.
Boundaries (Moore), 550, 555.
Boundary layer problems (Weyl), 100.
BRAUN, A. C., AND LASKARIS, T. Tumor Formation by Attenuated Crown-Gall Bacteria in the Presence of Growth-Promoting Substances, 468.
BRENTON, V. K. See Payne-Gaposchkin, C., 496.
DE BRETTEVILLE, A. See Buerger, M. J., 526.
BUCHANAN, J. M. See Hastings, A. B., 478.
BUERGER, M. J. A New Fourier Series Technique for Crystal Structure Determination, 281.
——. The Characteristics of Soap Hemihydrate Crystals, 529.
BUERGER, M. J., SMITH, L. B., DE BRETTEVILLE, A., AND RYER, F. V. The Lower Hydrates of Soaps, 526.
BURKE, M. H. See Wilson, E. B., 361.
BURKHOLDER, P. R., AND MCVEIGH, I. Synthesis of Vitamins by Intestinal Bacteria, 285.
——. The Increase of B Vitamins in Germinating Seeds, 440.
BURRIS, R. H. See Wilson, P. W., 243.
B vitamins (Burkholder and McVeigh), 440.

CAMPBELL, D. H. See Pressman, D., 77.
Carbon dioxide (van Niel, Ruben, Carson, Kamen and Foster), 8; (van Niel, Thomas, Ruben and Kamen), 157.
Carbon, radioactive (van Niel, Ruben, Carson, Kamen and Foster), 8; (van Niel, Thomas, Ruben and Kamen) 157.
CARSON, S. F. See van Niel, C. B., 8.
CASPARI, E. See Dunn, L. C., 205.
CASTLE, W. E. Size Genes of Mice, 69.
Cations, rôle of intracellular (Hastings and Buchanan), 478.
Cell nuclei (Mirsky and Pollister), 344.
Cells, division in (Sinnott), 36.
Cellular metabolism (van Niel, Ruben, Carson, Kamen and Foster), 8.
Cepheids (Shapley and McKibben), 200; (Shapley), 501.
Chain process (Opatowski), 83.
Cherry, Montmorency (Gustafson), 131.
CHILD, C. M. Oxidation-Reduction Patterns in Amphibian and Teleost Development, 339.
Chromosomal aberrations (Sax), 229.
Chromosome degeneration (Jones), 38.
Chromosomes (Wipf and Shackelford), 265; (Sutton), 268; (McClintock), 458.

Chromosomes, anaphase movement (Rhoades and Vilkomerson), 433.
Chromosomes, heterobrachial (Sparrow) 463.
Chromosomes, X-ray and neutron effects (Marshak), 29.
Ciass of surfaces (Fubini), 178.
CLOOS, E. Distortion of Stratigraphic Thicknesses Due to Folding, 401.
Combinatory Theory of Quantification (Curry), 564.
Complement fixation (Pressman, Campbell and Pauling), 77.
Complement-release reaction (Tyler), 391.
Confidence intervals (Wilson), 88.
Congruences (Vandiver), 144, 324.
Contingency tables (Wilson), **94**; (Wilson and Worcester), 378.
Continua (Moore), 544, 550.
Cotton oxidized (Heidelberger and Hobby), 516.
CRAIG, R. A. See Shapley, H., 192.
Crown gall-bacteria (Braun and Laskaris), 468.
Crystals, soap hemihydrate (Buerger), 529.
Crystal structure determination (Buerger), 281.
Cultures, toxicity of (Dubos, Hoberman and Pierce), 453.
CURRY, H. B. Some Advances in the Combinatory Theory of Quantification, 564.
Curvature, initial (Kasner and Mittleman), 48.
Curve, epidemic (Wilson and Burke), 361.
Curves, analytic (Weyl and Weyl), 417.

DARLINGTON, C. D., AND DOBZHANSKY, T. Temperature and "Sex-Ratio" in *Drosophila Pseudoöbscura*, 45.
DE CICCO, J. See Kasner, E., 52, 328, 413.
Degeneration, chromosome (Jones), 38.
Depolarization of neutron beams (Halpern and Holstein), 112.
Derivatives of a function (Schaeffer), 62.
Developing fruits (Sinnott), 36.
Diameters of galaxies (Shapley), 186.
Differential equations (Weyl), 100.
Differential equations: $y''' = Gy'' + Hy''^2$ (Kasner), 333.
Differentiation, formal exponential (Vandiver), 24.
Diffusion of gene products (Sax), 303.
Dihedral groups (Miller), 368.
Diiodotyrosine (Johnson and Tewkesbury), 73.
Dirichlet problem (Hadamard), 258.
Dissociation constant (Wilson, Burris and Lind), 243.
Distortion of stratigraphic thicknesses (Cloos), 401.
Division, rates of cell (Sinnott), 36.
DOBZHANSKY, T. See Darlington, C. D., 45.
Domains with compact boundaries (Moore), 555.
Drosophila Melanogaster (Fano), 119.
Drosophila Pseudoöbscura (Darlington and Dobzhansky), 45.
Dual families (Kasner and De Cicco), 52,
DUBOS, R. J., HOBERMAN, H. D., AND PIERCE, C. Some Factors Affecting the Toxicity of Cultures of *Shigella dysenteriae*, 453.
DUNN, L. C., AND CASPARI, E. Close Linkage between Mutations with Similar Effects, 205.
Dynamical trajectories (Kasner and Mittleman), 48.
Dynamics, inverse problem of (Kasner and De Cicco), 413.

Eclipsing binary systems (Kopal), 133.
Effectors, white pigmentary (Fries), 396.
Egg, avian (Romanoff and Bless), 306.
Egg, toad's (Painter and Taylor), 311.
EILENBERG, S., AND MACLANE, S. Natural Isomorphisms in Group Theory, 537.
Electron microscope (Luria and Anderson), 127.
Energy equation (Stewart), 161.
Enzymes (Wildman and Gordon), 217.
Epidemic curve (Wilson and Burke), 361.
Equations, differential (Weyl), 100; (Kasner), 333.
Euclidean 3-space (Kakutani), 16.
Expansions, generalized biorthogonal (Tseng), 170.
Extragalactic studies (Shapley), 186; (Shapley, McKibben and Craig), 192; (Shapley and McKibben), 200; (Shapley), 501.

Families, isothermal and dual (Kasner and De Cicco), 52.
FANKHAUSER, G., AND WATSON, R. C. Heat-Induced Triploidy in the Newt, *Triturus Viridescens*, 436.
FANO, U. An Autosomal Recessive Factor Inducing Semi-Sterility in *Drosophila Melanogaster* Females, 119.
Fermat's last theorem (Vandiver), 144.
Fertilizin of sea-urchins (Tyler), 391.
Fixation, complement (Pressman, Campbell and Pauling), 77.
Flows, measurable (Kakutani), 16.
Fluid, viscous compressible (Stewart), 161
Folding, stratigraphic (Cloos), 401.
FOSTER, J. W. See van Niel, C. B., 8.
Fourier series (Buerger), 281.
FRIES, E. F. B. White Pigmentary Effectors (*Leucophores*) in Killifishes. 396.
Frobenius's theorem (Miller), 251.
Fruitfulness in Montmorency cherry (Gustafson), 131.

Fruits, developing (Sinnott), 36.
FUBINI, G. On a Class of Surfaces, 178.
Function, derivatives of (Schaeffer), 62.
Fungi, growth of (Robbins), 352.
Fusion, nuclear (McClintick), 458.

Galactic studies (Shapley), 186; (Shapley, McKibben and Craig), 192; (Shapley and McKibben), 205; (Shapley), 501.
Galaxies, spiral (Shapley), 186.
GAPOSCHKIN, S. See Payne-Gaposchkin, C., 482, 490.
GARCIA, G. Generalization of Sundman's Fundamental Equality to the Case of More Than Three Bodies, 425.
——. Generalization of the Inequality of Sundman to the Case of More Than Three Bodies and to the Case of a Gravitational Gas, 428.
Gas, gravitational (Garcia), 428.
Gene products, diffusion of (Sax), 303.
Genes, of mice (Castle), 69.
Genetic control (Tatum and Beadle), 234.
Genetic nature of x-ray induced changes in pollen (Rick), 518.
Germinating seeds (Burkholder and McVeigh), 440.
Glasses, two-component (Lukesh), 277.
Glycogen formation (Hastings and Buchanan), 478.
GORDON, S. A. See Wildman, S. G., 217.
Gravitational gas (Garcia), 428.
Group theory (Eilenberg and MacLane), 537.
Groups (Miller), 80, 141, 368, 407.
Groups, monothetic (Halmos and Samelson), 254.
Groups, transformation (Smith), 293.
Growth (Jones), 38.
Growth of fungi (Robbins), 352.
Growth hormone (Gustafson), 131.
Growth inhibition (Bonner), 321.
Growth-promoting substances (Braun and Laskaris), 468.
Growth requirements of *Spirillum Serpens* (Pennington), 272.
Growth substances for *phycomyces* (Robbins and Kavanagh), 4, 65.
Guanine (Robbins and Kavanagh), 4.
GUSTAFSON, F. G. Concentration of Growth Hormone and Fruitfulness in the Montmorency Cherry, 131.

HADAMARD, J. On the Dirichlet Problem for the Hyperbolic Case, 258.
HALMOS, P. R., AND SAMELSON, H. On Monothetic Groups, 254.
HALPERN, O., AND HOLSTEIN, T. On the Depolarization of Neutron Beams by Magnetic Fields, 112.
Haptenic groups (Pressman, Campbell and Pauling), 77.
HARRISON, G. The Structure of Algebraic Moduls, 410.
HASTINGS, A. B., AND BUCHANAN, J. M. The Rôle of Intracellular Cations on Liver Glycogen Formation *in Vitro*, 478.
Heat-induced triploidy (Fankhauser and Watson), 436.
HEIDELBERGER, M., AND HOBBY, G. M. Oxidized Cotton, an Immunologically Specific Polysaccharide, 516.
Hemihydrate crystals of soap (Buerger), 529.
Hereditary nervous disorder (Sawin, Anders and Johnson), 123.
Heterobrachial chromosomes (Sparrow), 463.
HILLE, E. Representation of One-Parameter Semi-Groups of Linear Transformations, 175.
——. On the Analytical Theory of Semi-Groups, 421.
HOBBY, G. M. See Heidelberger, M., 516.
HOBERMAN, H. D. See Dubos, R. J., 453.
HOLSTEIN, T. See Halpern, O., 112.
Homogeneous measure algebras (Maharam), 108.
Hormone, growth (Gustafson), 131.
Hormones, sexual in Achyla (Raper), 509.
HUDSON, G. E. See Bartlett, N. R., 289.
Hybrid vigor (Jones), 38.
Hydra cluster of nebulae (Zwicky), 150.
Hydrates of soap (Buerger, Smith, de Bretteville and Ryer), 526.
Hyperbolic case (Hadamard), 258.
Hypergeometric polynomial (Bateman), 374.
Hypoxanthine (Robbins and Kavanagh), 65.

Immunologically specific polysaccharide (Heidelberg and Hobby), 516.
Inactivation of tobacco mosaic virus (Marshak and Takahashi), 211.
Inequality of Sundman (Garcia), 428.
Inhibition, reversible growth (Bonner), 321.
Initial curvature of dynamical trajectories (Kasner and Mittleman), 48.
Integers, sets of (Salem and Spencer), 561.
Integral, Laplace (Boas), 21.
Intensity, light (Bartlett and Hudson), 289.
Intersecting continua (Moore), 544.
Intestinal bacteria (Burkholder and McVeigh), 285.
Intracellular cations (Hastings and Buchanan), 478.
Inverse problem (Opatowski), 83.
Inverse problem of dynamics (Kasner and De Cicco), 413.
Inversion of a generalized Laplace integral (Boas), 21.
Isogonal trajectories (Kasner and De Cicco), 328.

Isomorphisms, natural (Eilenberg and MacLane), 537.
Isothermal families (Kasner and De Cicco), 52, 328.

Jeannerettia (Papenfuss), 446.
JOHNSON, R. B. See Sawin, P. B., 123.
JOHNSON, T. B., AND TEWKESBURY, L. B. The Oxidation of 3,5-Diiodotyrosine to Thyroxine, 73.
JONES, D. F. Chromosome Degeneration in Relation to Growth and Hybrid Vigor, 38.

KAKUTANI, S. Representation of Measurable Flows in Euclidean 3-Space, 16.
KAMEN, M. D. See van Niel, C. B., 8, 157.
KASNER, E. Differential Equations of the Type: $y''' = Gy'' + Hy''^2$, 333.
KASNER, E., AND DE CICCO, J. Generalized Transformation Theory of Isothermal and Dual Families, 52.
——. Transformation Theory of Isogonal Trajectories of Isothermal Families, 328.
——. Synthetic Solution of the Inverse Problem of Dynamics, 413.
KASNER, E., AND MITTLEMAN, D. A. General Theorem of the Initial Curvature of Dynamical Trajectories, 48.
KAVANAGH, F. See Robbins, W. J., 4, 65.
Killifishes (Fries), 396.
KOPAL, Z. The Calculation of Rotation Factors for Eclipsing Binary Systems, 133.

Laplace integral (Boas), 21.
LASKARIS, T. See Braun, A. C., 468.
Leaf proteins (Wildman and Gordon), 217.
Leucophores (Fries), 396.
Light intensity (Bartlett and Hudson), 289.
LIND, C. J. See Wilson, P. W., 243.
Linear transformations (Hille), 175.
Linkage, close (Dunn and Caspari), 205.
Liver glycogen formation (Hastings and Buchanan), 478.
Log $\Gamma(x)$ (Uhler), 59.
LUKESH, J. S. The Distribution of Metallic Atoms in Two-Component Glasses 277.
LURIA, S. E., AND ANDERSON, T. F. The Identification and Characterization of Bacteriophages with the Electron Microscope, 127.

MCCLINTOCK, B. The Fusion of Broken Ends of Chromosomes Following Nuclear Fusion, 458.
MCKIBBEN, V. See Shapley, H., 192, 200.
MACLANE, S. See Eilenberg, S., 537.
MCVEIGH, I. See Burkholder, P. R., 285, 440.
Magellanic cloud (Shapley and McKibben), 200.
Magnetic fields (Halpern and Holstein), 112.
Magnitude dispersion (Shapley, McKibben and Craig), 192.
MAHARAM, D. On Homogeneous Measure Algebras, 108.
MARSHAK, A. Relative Effects of X-Rays and Neutrons on Chromosomes in Different Parts of the "Resting Stage," 29.
MARSHAK, A., AND TAKAHASHI, W. N. The Effect of pH on Inactivation of Tobacco Mosaic Virus by X-Rays, 211.
Material ejected from novae (Payne-Gaposchkin and Gaposchkin), 482.
Measurable flows (Kakutani), 16.
Melanophores, sensitization of (Parker), 164.
MENGER, K. Statistical Metrics, 535.
Mesotrema (Papenfuss), 446.
Metabolism, cellular (van Niel, Ruben, Carson, Kamen and Foster), 8.
Metallic atoms (Lukesh), 277.
Metric spaces (Tseng), 170.
Metrics, statistical (Menger), 535.
Mice, size genes of (Castle), 69.
Microscope, electron (Luria and Anderson), 127.
MILLER, G. A. Maximal Sylow Subgroups of a Given Group, 80.
——. Certain Direct Products of the Groups of Self-Isometries, 141.
——. Some Deductions from Frobenius's Theorem, 251.
——. Automorphisms of the Dihedral Groups, 368.
——. The Permutation Groups of a General Degree, 407.
Minimal surfaces (Shiffman), 103.
MIRSKY, A. E., AND POLLISTER, A. W. Nucleoproteins of Cell Nuclei, 344.
MITTLEMAN, D. See Kasner, E., 48.
Moduls, algebraic (Harrison), 410.
Monothetic groups (Halmos and Samelson), 254.
MOORE, R. L. Concerning Separability, 56.
——. Concerning Intersecting Continua, 544.
——. Concerning a Continuum and Its Boundary, 550.
——. Concerning Domains Whose Boundaries Are Compact, 555.
Montmorency cherry (Gustafson), 131.
Mosaic, tobacco (Marshak and Takahashi), 211.
Mosquitoes (Sutton), 268.

Mutant, aminobenzoicless (Tatum and Beadle), 234.
Mutations (Dunn and Caspari), 205.

Nebulae (Zwicky), 150, 317, 355.
Nerve cutting in melanophores (Parker), 164.
Neurospora (Tatum and Beadle), 234.
Neutralization of anticomplementary action (Tyler), 391.
Neutron beams, depolarization of (Halpern and Holstein), 112.
Neutrons, effects on chromosomes (Marshak), 29.
Newt (Fankhauser and Watson), 436.
VAN NIEL, C. B., RUBEN, S., CARSON, S. F., KAMEN, M. D., AND FOSTER, J. W. Radioactive Carbon as an Indicator of Carbon Dioxide Utilization. VIII. The Rôle of Carbon Dioxide in Cellular Metabolism, 8.
VAN NIEL, C. B., THOMAS, J. O., RUBEN, S., AND KAMEN, M. D. Radioactive Carbon as an Indicator of Carbon Dioxide Utilization. IX. The Assimilation of Carbon Dioxide by Protozoa, 157.
Nitrogen fixation (Wilson, Burris and Lind), 243.
Nomenclature, algal (Papenfuss), 446.
Novae, material ejected from (Payne-Gaposchkin and Gaposchkin), 482.
Nuclear fusion (McClintock), 458.
Nuclei, cell (Mirsky and Pollister), 344.
Nucleic acid (Painter and Taylor), 311.
Nucleoproteins of cell nuclei (Mirsky and Pollister), 344.
Numbers, Bernoulli (Vandiver), 324.

One-parameter semi-groups (Hille), 175.
OPATOWSKI, I. An Inverse Problem Concerning a Chain Process, 83.
Orthogonal property (Bateman), 374.
Oscillations of derivatives (Schaeffer), 62.
OSGOOD, C. Prehistoric Contact between South America and the West Indies, 1.
Oxidation of 3,5-diiodotyrosine to thyroxine (Johnson and Tewkesbury), 73.
Oxidation-reduction (Child), 339.
Oxidized cotton (Heidelberger and Hobby), 516.

PAINTER, T. S., AND TAYLOR, A. N. Nucleic Acid Storage in the Toad's Egg, 311.
PAPENFUSS, G. F. Notes on Algal Nomenclature: *Pollexfenia*, *Jeanneretia* and *Mesotrema*, 446.
PARKER, G. H. Sensitization of Melanophores by Nerve Cutting, 164.
PAYNE-GAPOSCHKIN, C., AND BRENTON, V. K. A Study of the RV Tauri Variables, 496.
PAYNE-GAPOSCHKIN, C., AND GAPOSCHKIN, S. On the Dimensions and Constitution of Variable Stars, 490.
——. On the Material Ejected from Novae, 482.
Patterns of oxidation-reduction (Child), 339.
PAULING, L. See Pressman, D., 77.
PENNINGTON, D. E. Purines as Growth Requirements of *Spirillum Serpens*, 272.
Periods for 343 Cepheids (Shapley and McKibben), 200.
Period-luminosity relation (Shapley, McKibben and Craig), 192.
Permutation groups (Miller), 407.
Perseus cluster of nebulae (Zwicky), 317, 355.
Photographic amplitudes (Shapley), 501.
Phycomyces (Robbins and Kavanagh), 4, 65.
PIERCE, C. See Dubos, R. J., 453.
Pollen, x-ray induced changes in (Rick), 518.
Pollexfenia (Papenfuss), 446.
POLLISTER, A. W. See Mirsky, A. E., 344.
Polynomial, hypergeometric (Bateman), 374.
Polysaccharide, immunologically specific (Heidelberger and Hobby), 516.
Potentials, bioelectric (Romanoff and Bless), 306.
Prehistoric contact (Osgood), 1.
PRESSMAN, D., CAMPBELL, D. H., AND PAULING, L. Complement Fixation with Simple Substances Containing Two or More Haptenic Groups, 77.
Progression, arithmetical (Salem and Spencer), 561.
Proteins, isolated leaf (Wildman and Gordon), 217.
Protozoa (van Niel, Thomas, Ruben and Kamen), 157.
Purines (Pennington), 272.
Pyrimidine analog (Robbins), 352.

Quantification, Combinatory Theory of (Curry), 564.

Rabbit, hereditary nervous disorder (Sawin, Anders and Johnson), 123.
Radioactive carbon (van Niel, Ruben, Carson, Kamen and Foster), 8; (van Niel, Thomas, Ruben and Kamen), 157.
RAPER, J. R. Sexual Hormones in Achlya. V. Hormone A′, A Male-Secreted Augmenter or Activator of Hormone A, 509.
Recessive factor, autosomal (Fano), 119.
Rectifiable boundary (Shiffman), 103.
Red fox, chromosomes of (Wipf and Shackelford), 265.

Reduction, oxidation (Child), 339.
Responses, visual (Bartlett and Hudson), 289.
Reversible growth inhibition (Bonner), 321.
RHOADES, M. M., AND VILKOMERSON, H. On the Anaphase Movement of Chromosomes, 433.
RICK, C. M. The Genetic Nature of X-Ray Induced Changes in Pollen, 518.
Rings (Vandiver), 24.
ROBBINS, W. J. A Pyrimidine Analog of Thiamine and the Growth of Fungi, 352.
ROBBINS, W. J., AND KAVANAGH, F. Guanine and Factor Z_1, Growth Substances for *Phycomyces*, 4.
——. Hypoxanthine, a Growth Substance for *Phycomyces*, 65.
ROMANOFF, A. L., AND BLESS, A. A. The Effect of X-Ray Stimulation on the Bioelectric Potentials of the Avian Egg, 306.
Roots, tomato (Bonner), 321.
Rotation factors for binaries (Kopal), 133.
RV Tauri variables (Payne-Gaposchkin and Brenton), 496.
RUBEN, S. See van Niel, C. B., 8, 157.
RYER, F. V. See Buerger, M. J., 526.

SALEM, R., AND SPENCER, D. C. On Sets of Integers Which Contain No Three Terms in Arithmetical Progression, 561.
Salivary gland type chromosomes (Sutton), 268.
SAMELSON, H. See Halmos, P. R., 254.
SAWIN, P. B., ANDERS, M. V., AND JOHNSON, R. B. "Ataxia," a Hereditary Nervous Disorder of the Rabbit, 123.
SAX, K. The Distribution of X-Ray Induced Chromosomal Aberrations, 229.
——. Diffusion of Gene Products, 303.
SCHAEFFER, A. C. Oscillations of the Derivatives of a Function, 62.
Sea-urchin fertilizin (Tyler), 391.
Seeds, germinating (Burkholder and McVeigh), 440.
Self-isometries (Miller), 141.
Semi-groups, one-parameter (Hille), 175, 421.
Semi-sterility in *Drosophila Melanogaster* females (Fano), 119.
Sensitization of melanophores (Parker), 164.
Separability (Moore), 56.
"Sex-Ratio" in *Drosophila Pseudoöbscura* (Darlington and Dobzhansky), 45.
Sexual hormones in Achlya (Raper), 509.
SHACKELFORD, R. M. See Wipf, L., 265.
SHAPLEY, H. Galactic and Extragalactic Studies, XIII. Note on the Comparative Diameters of Spheroidal and Spiral Galaxies, 186.
——. Galactic and Extragalactic Studies, XVI. Photographic Amplitudes of Classical Cepheids, 501.
SHAPLEY, H., AND MCKIBBEN, V. Galactic and Extragalactic Studies, XV. On the Distribution of Periods for 343 Cepheids in the Small Magellanic Cloud, 200.
SHAPLEY, H., MCKIBBEN, V., AND CRAIG, R. A. Galactic and Extragalactic Studies, XIV. On the Magnitude Dispersion in the Period-Luminosity Relation, 192.
SHIFFMAN, M. Unstable Minimal Surfaces with any Rectifiable Boundary, 103.
Shigella dysenteriae, cultures of (Dubos, Hoberman and Pierce), 453.
SINNOTT, E. W. Comparative Rates of Division in Large and Small Cells of Developing Fruits, 36.
Size genes of mice (Castle), 69.
SMITH, L. B. See Buerger, M. J., 526.
SMITH, P. A. Stationary Points of Transformation Groups, 293.
Soap (Buerger), 529; (Buerger, Smith, de Bretteville, Ryer), 526.
South America (Osgood), 1.
Spaces, unitary and metric (Tseng), 170.
SPARROW, A. H. Non-Random Uncoiling in Heterobrachial Chromosomes, 463.
SPENCER, D. C. See Salem, R., 561.
Spinach (Wildman and Gordon), 217.
Spiral galaxies (Shapley), 186.
Spirillum Serpens (Pennington), 272.
Stars, variable (Payne-Gaposchkin and Gaposchkin), 490.
Stationary points (Smith), 293.
Statistical metrics (Menger), 535.
STEWART, H. J. The Energy Equation for a Viscous Compressible Fluid, 161.
Stirling's series for log $\Gamma(x)$ (Uhler), 59.
Storage of nucleic acid (Painter and Taylor), 311.
Stratigraphic thicknesses, distortion of (Cloos), 401.
Structure, crystal (Buerger), 281.
Structure of algebraic moduls (Harrison), 410.
Subgroups, maximal sylow (Miller), 80.
Substances, growth-promoting (Braun and Laskaris), 468.
Sundman, generalization of (Garcia), 425, 428.
Surfaces, class of (Fubini), 178.
Surfaces, unstable minimal (Shiffman), 103.
SUTTON, E. Salivary Gland Type Chromosomes in Mosquitoes, 268.
Sylow subgroups (Miller), 80.
Synthetic solution (Kasner and De Cicco), 413.

TAKAHASHI, W. N. See Marshak, A., 211

TATUM, E. L., AND BEADLE, G. W. Genetic Control of Biochemical Reactions in *Neurospora:* an "Aminobenzoicless" Mutant, 234.
TAYLOR, A. N. See Painter, T. S., 311.
Teleost development (Child), 339.
Temperature in *Drosophila Pseudoöbscura* (Darlington and Dobzhansky), 45.
TEWKESBURY, L. B. See Johnson, T. B., 73.
Theory of Quantification (Curry), 564.
Thiamine (Robbins), 352.
THOMAS, J. O. See van Niel, C. B., 157.
Tobacco mosaic (Marshak and Takahashi), 211.
Thyroxine (Johnson and Tewkesbury), 73.
Toad's egg (Painter and Taylor), 311.
Tomato roots, isolated (Bonner), 321.
Toxicity of cultures (Dubos, Hoberman and Pierce), 453.
Trajectories, isogonal (Kasner and De Cicco), 328.
Trajectories, dynamical (Kasner and Mittleman), 48.
Transformation groups (Smith), 293.
Transformation theory (Kasner and De Cicco), 52, 328.
Transformations, linear (Hille), 175.
Triploidy, heat-induced (Fankhauser and Watson), 436.
Triturus Viridescens (Fankhauser and Watson), 436.
TSENG, Y. Y. On Generalized Biorthogonal Expansions in Metric and Unitary Spaces, 170.
t-Test (Wilson and Worcester), 297.
Tumor formation (Braun and Laskaris), 468.
Two-component glasses (Lukesh), 277.
TYLER, A. A Complement-Release Reaction: The Neutralization of the Anticomplementary Action of Sea-Urchin Fertilizin by Antifertilizin, 391.

UHLER, H. S. The Coefficients of Stirling's Series for Log $\Gamma(x)$, 59.
Uncoiling in heterobrachial chromosomes (Sparrow), 463.
Unitary spaces (Tseng), 170.

VANDIVER, H. S. On Formal Exponential Differentiation in Rings, 24.
——. Certain Congruence Criteria Connected with Fermat's Last Theorem 144.
——. General Congruences Involving the Bernoulli Numbers, 324.
Variable stars (Payne-Gaposchkin and Gaposchkin), 490.
Variables, RV Tauri (Payne-Gaposchkin and Brenton), 496.
Vigor, hybrid (Jones), 38.
VILKOMERSON, H. See Rhoades, M. M., 433.
Virus, tobacco mosaic (Marshak and Takahashi), 211.
Viscous compressible fluid (Stewart), 161.
Visual responses (Bartlett and Hudson), 289.
Vitamins (Burkholder and McVeigh), 285 440.

WATSON, R. C. See Fankhauser, G., 436.
West Indies (Osgood), 1.
WEYL, H. Concerning the Differential Equation of Some Boundary Layer Problems, II, 100.
WEYL, H., AND WEYL, J. On the Theory of Analytic Curves, 417.
WEYL, J. See Weyl, H., 417.
White pigmentary effectors (Fries), 396.
WILDMAN, S. G., AND GORDON, S. A. The Release of Auxin from Isolated Leaf Proteins of Spinach by Enzymes, 217.
WILSON, E. B. On Confidence Intervals, 88.
——. On Contingency Tables, 94.
WILSON, E. B., AND BURKE, M. H. The Epidemic Curve, 361.
WILSON, E. B., AND WORCESTER, J. Note on the *t*-Test, 297.
WILSON, E. B., AND WORCESTER, J. Contingency Tables, 378.
——. The Association of Three Attributes, 384.
WILSON, P. W., BURRIS, R. H., AND LIND C. J. The Dissociation Constant in Nitrogen Fixation by *Azotobacter*, 243.
WIPF, L., AND SHACKELFORD, R. M. Chromosomes of the Red Fox, 265.
WORCESTER, J. See Wilson, E. B., 297, 378, 384.

X-rays (Marshak), 29; (Marshak and Takahasi), 211; (Sax), 229; (Romanoff and Bless), 306; (Rick), 518.

$y''' = Gy'' + Hy'^2$ (Kasner), 333.

ZWICKY, F. On the Physical Characteristics of the Hydra Cluster of Nebulae, 150.
——. On the Perseus Cluster of Nebulae, 317.
——. On the Physical Characteristics of the Perseus Cluster of Nebulae, 355.

F